Exact Renormalization for the Gross-Neveu Model of Quantum Fields

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We show that the Wilson renormalization-group flow of effective Lagrangians in fermionic models can be explicitly controlled using convergent perturbation expansions. A simple rigorous construction of two-dimensional asymptotically free field theories (the Gross-Neveu models) results.

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It is an old observation[1] that in QED, for a finite ultraviolet (uv) and infrared (ir) cutoff, the standard perturbation expansion (PE) is a convergent series. This result easily extends to an arbitrary purely fermionic (local) interaction in sharp contrast with bosonic theories where even in the presence of cutoffs, the PE is only asymptotic, its convergence being obstructed by instantons.[2] The simplicity of fermions is due to the Pauli principle: With uv cutoff there is effectively a finite number of modes per unit volume which can be only singly excited, and hence there is no local number divergence in contrast to bosons. However, as the uv cutoff is removed and renormalization performed, this property seems to be lost. Indeed, in renormalizable models the divergence of the renormalized PE seems to emerge as signaled by the renormalon singularities of its Borel transform.[3] It is due to the factorial growth of the amplitudes of certain graphs.

In the present paper, we describe an approach to the renormalization of fermionic theories which allows rigorous construction of the renormalizable asymptotically free models by using only convergent expansions.[4] We show that for the Gross-Neveu models in two dimensions,[5] once the uv cutoff is successively lowered using Wilson's renormalization-group (RG) idea,[6] the resulting effective Lagrangians may be computed from each other by means of convergent PE's and are given by convergent power series in the fields. Thus we are able to evaluate the full Wilson RG flow in the space of all actions close to the massless Gaussian one. The analysis extends to the Schwinger functions providing a rigorous construction of quantum field theory which is renormalizable and asymptotically free but not superrenormalizable.

We take the bare Euclidean action ($\psi^{\ddagger}=\psi$ or $\bar\psi$)

$$S(\psi^{\ddagger},\Lambda)=S_0(\psi^{\ddagger},\Lambda)+\tilde S(\psi^{\ddagger},\Lambda), \tag{1}$$

where

$$S_0(\psi^{\ddagger},\Lambda)=\int\bar\psi\, i\boldsymbol{\partial}_\Lambda\psi \tag{2}$$

with the free cutoff propagator in the momentum space

$$(i\boldsymbol{\partial}_\Lambda)^{-1}(p)=(\mathbf{p}/p^2)\exp[-p^2/\Lambda^2], \tag{3}$$

and

$$\tilde S(\psi^{\ddagger},\Lambda)=z(\Lambda)\int\bar\psi\, i\boldsymbol{\partial}\psi-g(\Lambda)\int(\bar\psi\psi)^2. \tag{4}$$

$\psi^{\ddagger}$ carries $N>1$ flavors and (1) has the U(N) symmetry.[7] The effective low-momentum actions are defined by integrating out fluctuations with momenta between Λ and $\tilde\Lambda$:

$$\exp[-S_{\tilde\Lambda}^{\text{eff}}(\psi^{\ddagger},\Lambda)]=\text{const}\times\exp\Big[-\int\bar\psi\, i\boldsymbol{\partial}_{\tilde\Lambda}\psi\Big]\int\exp[-\tilde S(\psi^{\ddagger}+\zeta^{\ddagger},\Lambda)-\langle\bar\zeta|\mathscr{T}^{-1}|\zeta\rangle]D\bar\zeta\, D\zeta, \tag{5}$$

where in the momentum space

$$\mathscr{T}(p)=(\mathbf{p}/p^2)(\exp[-p^2/\Lambda^2]-\exp[-p^2/\tilde\Lambda^2]). \tag{6}$$

In the standard perturbative approach one expands in the renormalized coupling g

$$S_{\tilde\Lambda}^{\text{eff}}(\psi^{\ddagger},\Lambda)\sim\sum_{p=0}^{\infty}g^p\sigma_{\tilde\Lambda}^{p}(\psi^{\ddagger},\Lambda) \tag{7}$$

and the perturbative renormalization theory tells us that $\sigma_{\tilde\Lambda}^{p}(\psi^{\ddagger},\Lambda)$ have limits as $\Lambda\to\infty$ if $g(\Lambda)$ and $z(\Lambda)$ are chosen appropriately as a formal power series in g.[8] The resulting expansion (7) is expected to diverge, however.

What we show is that, provided the bare couplings are chosen as

$$\frac{1}{g(\Lambda)}=\frac{1}{g}-\beta_2\ln\frac{\Lambda}{\mu}+\frac{\beta_3}{\beta_2}\ln\left(1-\beta_2 g\ln\frac{\Lambda}{\mu}\right), \tag{8}$$

$$z(\Lambda)=-(\gamma_2/\beta_2)g(\Lambda), \tag{9}$$

with $g>0$ small and $\beta_2<0$, β_3 and γ_2 being the standard coefficients of the perturbative β and γ functions, then for $\tilde{\Lambda}\geqslant\mu$ ($\tilde{\psi}=\psi^\ddagger$ or $\partial\psi^\ddagger$),

$$S_{\tilde{\Lambda}}^{\text{eff}}(\psi^\ddagger,\Lambda)=\sum_{m=2}^{\infty}S_{\tilde{\Lambda}}^{m}(x_1,\ldots,x_m,\Lambda)\prod_{i=1}^{m}\tilde{\psi}(x_i)\,dx_i, \tag{10}$$

where the series converges (see below what this means) uniformly in Λ and $S_{\tilde{\Lambda}}^{m}(x,\Lambda)$ has the limit $\Lambda\to\infty$. Perturbatively, $S_{\tilde{\Lambda}}^{m}$ corresponds to all the graphs of (7) with m external legs and as a function of g has an expansion in powers of g which becomes divergent as $\Lambda\to\infty$.

We obtain S^m in a different way, however. Instead of performing the full functional integral in (5) at once, we follow the RG idea[6] and reduce the cutoff little by little. Take $L=O(1)$ and compute iteratively

$$S(\Lambda)\to S_{\Lambda/L}^{\text{eff}}(\Lambda)\to S_{\Lambda/L^2}^{\text{eff}}(\Lambda)\to\ldots\to S_{\Lambda/L^n}^{\text{eff}}, \tag{11}$$

with $\Lambda=L^n\tilde{\Lambda}$. After the translation to dimensionless variables,

$$H_{\tilde{\Lambda}}^{\text{eff}}(\psi^\ddagger)=S_{\tilde{\Lambda}}^{\text{eff}}(\tilde{\Lambda}^{1/2}\psi^\ddagger(\tilde{\Lambda}\,\cdot)), \tag{12}$$

rewrite (5) as

$$H_{\tilde{\Lambda}}^{\text{eff}}(\psi^\ddagger,\Lambda)=\int\bar{\psi}i\partial_1\psi+\tilde{H}_{\tilde{\Lambda}}^{\text{eff}}(\psi^\ddagger,\Lambda), \tag{13}$$

where iteratively

$$\tilde{H}_{\tilde{\Lambda}/L}^{\text{eff}}(\psi^\ddagger,\Lambda)=-\ln\int\exp[-\tilde{H}_{\tilde{\Lambda}}^{\text{eff}}(L^{-1/2}\psi^\ddagger(\cdot/L)+Z^\ddagger)-\langle\bar{Z}|\Gamma^{-1}|Z\rangle]D\bar{Z}\,DZ+\text{const} \tag{14}$$

with

$$\Gamma(p)=(\mathbf{p}/p^2)(\exp[-p^2]-\exp[-L^2p^2]). \tag{15}$$

Equation (14) is our RG transformation. It is computed perturbatively,

$$\tilde{H}_{\tilde{\Lambda}/L}^{\text{eff}}(\psi^\ddagger)=\sum_{p=1}^{\infty}\frac{(-1)^{p+1}}{p!}\langle(\tilde{H}_{\tilde{\Lambda}}^{\text{eff}})^p\rangle_\Gamma^c, \tag{16}$$

where $\langle\ \rangle_\Gamma^c$ denotes the connected expectation with the propagator Γ. For example, in the first step in (11) when $\tilde{H}_{\tilde{\Lambda}}^{\text{eff}}=z(\Lambda)\int\bar{\psi}i\boldsymbol{\partial}\psi-g(\Lambda)\int(\bar{\psi}\psi)^2$, (16) is given by the Feynman graphs with Γ lines. Note that such graphs are *finite*: Γ has the uv and ir cutoffs. The point is that (16) not only is finite term by term *but is a convergent expansion.*

To see why this is so, consider again the first step, put for simplicity $z(\Lambda)=0$, and compute

$$\tilde{H}_{\tilde{\Lambda}/L}^{\text{eff}}(0)=-\sum_{p=1}^{\infty}\frac{g(\Lambda)^p}{p!}\int dx_1\cdots dx_p\Big\langle\prod_{i=1}^{p}[\bar{Z}(x_i)Z(x_i)]^2\Big\rangle_\Gamma^c, \tag{17}$$

i.e., the sum of vacuum graphs with propagator Γ. The number of graphs in the pth term of (17) is $\sim(p!)^2$, i.e., potentially dangerous: In a bosonic theory (17) would indeed diverge as $\sum_p p!$. However, in the fermionic theory cancellations occur. Consider, e.g., a (disconnected) Green's function (ignore indices)

$$\Big\langle\prod_{i=1}^{p}Z(x_i)\bar{Z}(y_i)\Big\rangle_\Gamma=\sum_\pi(-1)^\pi\prod_i\Gamma(x_i-y_{\pi(i)})=\det[\Gamma(x_i-y_j)] \tag{18}$$

which is bounded by $(\text{const})^p$ by the Hadamard inequality if x_i,y_j are lattice points (Γ has exponential decay). This should be contrasted with the bosonic $p!$ behavior and is due to the exclusion principle: Only $2N$ x_i's may be identical since $Z_\alpha^i(x)^2=0$. Thus, many x_i,y_j are distant from each other and most terms in the sum in (18) are very small, leading to the $(\text{const})^p$ behavior. For x_i,y_j close but noncoinciding, a slightly more involved argument is needed.[4] In any case, the result is that the integral in (17) is bounded by volume times $C^p p!$ and (17) converges for $g(\Lambda)$ small (as is our case). The latter remains true for $\psi^\ddagger \neq 0$: (16) is also convergent (in an infinite Grassmann algebra where $\psi^\ddagger$ is a bounded operator). As a result

$$\tilde{H}^{\text{eff}}_{\Lambda/L}(\psi^\ddagger) = \sum_m \int H^m_{\Lambda/L}(x_1, \ldots, x_m) \prod_i \tilde{\psi}(x_i)\, dx_i, \tag{19}$$

where each $H^m_{\Lambda/L}$ is given in terms of convergent PE and satisfies geometric bounds [see (28) below]. These bounds guarantee in turn the convergence of (19) which is also crucial because it allows us to iterate the procedure.

To that end, we take $\tilde{H}^{\text{eff}}_{\tilde{\Lambda}}$ of the form (19) and evaluate $\tilde{H}^{\text{eff}}_{\tilde{\Lambda}/L}$ perturbatively as in (16). This will again be a convergent PE. The only difference with the first step is that the $Z^\ddagger$ integration involves arbitrary (nonlocal) vertices now. What results is a recursion for the vertex kernels H^m of the form

$$H^m_{\tilde{\Lambda}/L}(x_i) = L^{3m/2} H^m_{\tilde{\Lambda}}(Lx_i) + F(x_i, \{H^n_{\tilde{\Lambda}}\}), \tag{20}$$

where we have singled out the diagonal linear contribution [it would be just the whole linear one if we normal ordered the fields in (19) with respect to the free covariance]. Equation (20) is a nonlinear transformation in the space of vertex kernels $H^m_{\tilde{\Lambda}}$ but F is given in terms of a convergent power series in $H^n_{\tilde{\Lambda}}$'s. To see what kind of properties of H^m's survive the iteration of (20) note that after the first step $H^m_{\Lambda/L}$ are given in terms of connected graphs with Γ lines. The exponential falloff of $\Gamma(x)$ leads to an approximate locality of $H^m_{\Lambda/L}(x_i)$ up to exponentially decaying tails. It is useful to introduce the notation

$$\| H^m_{\tilde{\Lambda}} \| = \int dx_2 \cdots dx_m \, | H^m_{\tilde{\Lambda}}(x_1, \ldots, x_m) | \exp[\mathscr{L}(x_1, \ldots, x_m)], \tag{21}$$

where $\mathscr{L}(x_1, \ldots, x_m)$ is the length of the shortest tree graph on $x_1, \ldots, x_m$. Since higher m involve higher powers of the coupling constant, $\| H^m_{\Lambda/L} \|$ has a geometric bound. For iteration, note the leading part of (20) has a nice contractive property:

$$\| L^{3m/2} H^m_{\tilde{\Lambda}}(L\,\cdot) \| \leq L^{2-m/2} \| H^m_{\tilde{\Lambda}} \|, \tag{22}$$

i.e., all vertices but for $m=2$ and $m=4$ contract. Equation (22), of course, reflects the dimensionality of the corresponding operators. Hence, the idea is to separate the dangerous term, namely, the local parts of $m=2,4$. In fact, symmetries restrict the form of these local terms to those present in $S(\Lambda)$ (this is how renormalizability shows up in the present formulation). The picture we obtain is that there are two slowly varying couplings: $z_{\tilde{\Lambda}}$ of the kinetic term and $g_{\tilde{\Lambda}}$. The rest of H^{eff} stabilizes and flows with these couplings:

$$\tilde{H}^{\text{eff}}_{\tilde{\Lambda}}(\psi^\ddagger) = z_{\tilde{\Lambda}} \int \bar{\psi} i \partial \psi - g_{\tilde{\Lambda}} (\bar{\psi}\psi)^2 + \sum_m \int \hat{H}^m_{\tilde{\Lambda}}(x_i) \prod_i \tilde{\psi}(x_i)\, dx_i, \tag{23}$$

where for $m=2$ both fields carry derivatives and for $m=4$ at least one of them. Thus the sum over m runs over irrelevant operators whose kernels contract in the leading approximation to the RG transformation. Upon iteration of (23) new couplings $z_{\tilde{\Lambda}/L}, g_{\tilde{\Lambda}/L}$ and new vertex kernels $H^m_{\tilde{\Lambda}/L}$ arise which have convergent PE's in terms of the ones of $H^{\text{eff}}_{\tilde{\Lambda}}$.

The continuum limit is easy once we know the flow of the couplings in the lowest orders. As is well known, in the flow

$$\tilde{\Lambda} \frac{dg_{\tilde{\Lambda}}}{d\tilde{\Lambda}} = \beta_2 g^2_{\tilde{\Lambda}} + \beta_3 g^3_{\tilde{\Lambda}} + O(g^4_{\tilde{\Lambda}}) \tag{24}$$

only the coefficients β_2 and β_3 are responsible for the sensitive dependence on the initial conditions: Picking the bare coupling as in (8), the physical one g_μ will be $g + o(g^2)$ independent of the $O(g^4_{\tilde{\Lambda}})$ in (24). Thus our idea is to compute $\tilde{H}^{\text{eff}}_{\tilde{\Lambda}}$ exactly to the third order of $g_{\tilde{\Lambda}}$ and bound the remainder. Using the convergence of the PE for the step $\tilde{\Lambda} \to \tilde{\Lambda}/L$ and eliminating $z_{\tilde{\Lambda}}$ which is essentially driven by $g_{\tilde{\Lambda}}$,

$$\Lambda \frac{dz_{\tilde{\Lambda}}}{d\tilde{\Lambda}} = -\gamma_2 g^2_{\tilde{\Lambda}} + O(g^3_{\tilde{\Lambda}}), \tag{25}$$

we obtain (24) (in the discrete form). The upshot is that for all $\tilde{\Lambda}$, $\mu \leq \tilde{\Lambda} \leq \Lambda$,

$$z_{\tilde{\Lambda}} = -(\gamma_2/\beta_2) g_{\tilde{\Lambda}} + O(g^2_{\tilde{\Lambda}}), \tag{26}$$

$$g_{\tilde{\Lambda}} = g(\tilde{\Lambda}) + o(g(\tilde{\Lambda})^2), \tag{27}$$

and

$$\| \tilde{H}^m_{\tilde{\Lambda}} \| \leq \text{const} \times g^2_{\tilde{\Lambda}} \epsilon^m, \tag{28}$$

where ϵ is small for g small assuring the convergence of (23). Equation (26) to (28) show that the low-energy effective actions are bounded uniformly in the cutoff Λ. It is then an easy matter to show that they actually have the $\Lambda \to \infty$ limit.

In summary, to demonstrate the nonperturbative renormalizability of the model we have to deal with convergent well-defined PE's for uv and ir cutoff functional integrals (in particular, divergent diagrams are never encountered). Only few one- and two-loop graphs need to be computed; for the rest brute-force estimates are sufficient. The latter are easy exactly because the diagrams are finite and the PE converges.

In order to establish the existence of the $\Lambda \to \infty$ limit of the Green's functions, we repeat the above analysis with sources; e.g., for the two-point function we add to $S(\Lambda)$ a term $\eta_1 \psi^i_\alpha(x) + \eta_2 \bar{\psi}^j_\beta(y)$ to get $S^{\rm eff}_\Lambda(\psi^\ddagger, \eta_i, \Lambda)$. Then

$$\langle \psi(x)\bar{\psi}(y)\rangle_{S(\Lambda)} = \left\langle \frac{\partial^2 S^{\rm eff}_\mu}{\partial\eta_1\partial\eta_2}(\eta=0)\right\rangle_{S^{\rm eff}_\mu(\eta=0)}. \quad (29)$$

The right-hand side has a convergent PE in finite volume. To deal with the infrared question, we consider the model with an explicit mass term. The analysis generalizes to this case in a straightforward way and $S^{\rm eff}_\mu$ stays as before, only a mass term is present. Now the PE for the right-hand side of (29) will have a massive uv cutoff propagator and will converge also for infinite volume, providing a construction of the $\Lambda \to \infty$ limit of the Schwinger functions on the left-hand side.

At $N=\infty$, the massless model is known to exhibit chiral symmetry breaking and dynamic mass generation[5] (believed to survive for all $N > 1$[9]). We hope to be able to prove this rigorously for N large but finite, the idea being that the PE used to compute the RG transformations converges *uniformly in N.* Also, for large N, this may be used to study the nonrenormalizable $d=3$ theory believed to be governed by a nontrivial uv stable fixed point.[10] There is thus a good chance that a rigorous construction of a perturbatively nonrenormalizable quantum field theory is possible.

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