Variational Calculation of the Bound-State Wave Function in $:\lambda(\phi^6-\phi^4)_2:$

Jurij W. Darewych, Marko Horbatsch,[a] and Roman Koniuk
Department of Physics, York University, Toronto, Ontario M3J 1P3, Canada
(Received 6 February 1985)

The Gaussian variational approximation is used to calculate the two-particle bound-state binding energy and wave function in the model $:\lambda(\phi^6-\phi^4):$ in $1+1$ dimensions. An analytic result is compared to the perturbative calculation of Dimock and Eckmann and to the numerical, lattice work of Barnes and Daniell.

PACS numbers: 11.10.St, 11.10.Ef

The model of Glimm, Jaffe, and Spencer[1] possessing the Lagrangian

$$\tfrac{1}{2}(\partial_\mu\phi)^2 - :\lambda_B(\phi^6-\phi^4)+\tfrac{1}{2}m_B^2\phi^2: \tag{1}$$

in $1+1$ dimensions was introduced by the authors as a simple field theory with a bound state. The model has been studied perturbatively by Dimock and Eckmann[2] and numerically by Barnes and Daniell.[3] Both studies confirm the existence of a bound state at roughly the same mass for a given choice of λ_B. We have studied this model using the variational method and have obtained an approximate two-particle wave function and an expression for the bound-state mass.

The variational method is ideally suited for investigating the bound-state spectrum for all values of the parameters. The method has been used sparingly in the twenty years since its introduction into quantum field theory by Schiff.[4] A complete set of references is given by Stevenson[5] who recently has argued persuasively for its use in obtaining the Gaussian effective potential.

To compare with the numerical calculations we study the model normal ordered at the mass m_B. [We note that the non-normal-ordered $\lambda_B(\phi^6-\phi^4)$ theory does not have any bound-state solutions.] In practice this means that we are studying the potential

$$V(\phi)=\lambda_B\{\phi^6-[1+15I_0(m_B)]\phi^4+[6I_0(m_B)+45I_0^2(m_B)]\phi^2\}+\tfrac{1}{2}m_B^2\phi^2+\text{const}, \tag{2}$$

where

$$I_0(\mu)=\int_{-\infty}^{\infty}\frac{dp}{4\pi\omega(\mu,p)},\quad \omega^2(\mu,p)=\mu^2+p^2. \tag{3}$$

We note that the coefficient of ϕ^2 for the normal-ordered theory is not $\frac{1}{2}m_B^2$ and thus the bare mass is given by

$$m_B^2+12\lambda_B I_0(m_B)+90\lambda_B I_0^2(m_B).$$

The divergent integral $I_0(\mu)$ may be defined with an ultraviolet cutoff; however, it turns out that the calculated masses are completely independent of the cutoff.

We first compute the expectation value of the Hamiltonian sandwiched between a trial vacuum state which is just the free-field vacuum with a variational mass parameter Ω. The optimization condition on Ω gives the mass-gap equation

$$m_B^2-\Omega^2+\frac{6}{\pi}\lambda_B\ln\left(\frac{\Omega}{m_B}\right)+\frac{90}{4\pi^2}\lambda_B\ln^2\left(\frac{\Omega}{m_B}\right)=0. \tag{4}$$

In fact the solution to this equation, $\overline{\Omega}$, satisfies

$$\overline{\Omega}^2=m_R^2\equiv\left.\frac{d^2\mathscr{V}(\phi_0,\overline{\Omega})}{d\phi_0^2}\right|_{\phi_0=0}, \tag{5}$$

where $\mathscr{V}(\phi_0,\overline{\Omega})$ is the Gaussian effective potential.[6] The gap equation has two solutions. For values of $\lambda_B<\frac{1}{3}\pi\overline{\Omega}^2$ the solution

$$m_B=\overline{\Omega}=m_R \tag{6}$$

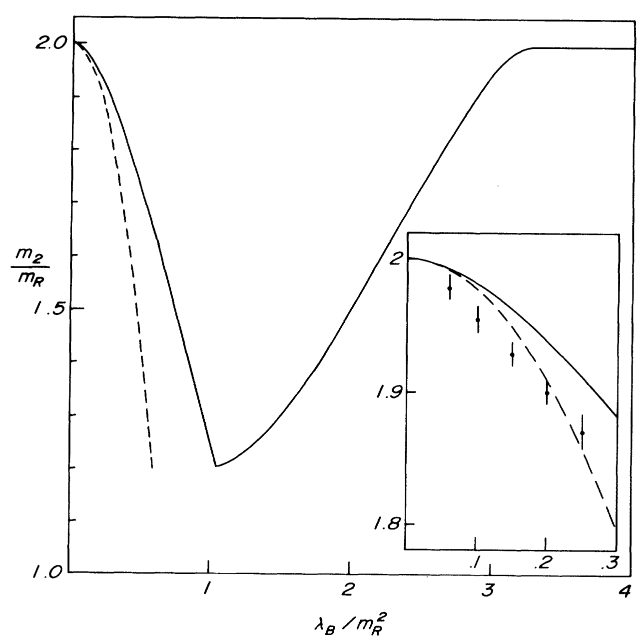

FIG. 1. The two-particle bound-state mass (m_2/m_R) vs coupling constant (λ/m_R^2). The solid curve is the present variational calculation. Perturbation theory gives the dashed curve (Ref. 2) and the points with error bars are the result of a numerical lattice calculation (Ref. 3).

corresponds to the minimum vacuum energy. For λ_B greater than this critical coupling the nontrivial solution of Eq. (4) minimizes the ground-state energy.

In both cases the same value of Ω minimizes the one- and two-particle energies. These states are obtained by acting on the vacuum with one or two creation operators, respectively. Thus our trial two-particle state is[4,6]

$$|2\rangle_\Omega = \int \frac{dp}{4\pi\omega(\Omega,p)}\sigma(p)a_\Omega^\dagger(p)a_\Omega^\dagger(-p)|0\rangle_\Omega. \tag{7}$$

Here $\sigma(p)$ is the Fourier transform of the two-particle bound-state wave function. The bound-state energy m_2 is therefore

$$m_2 \equiv \langle 2|H|2\rangle - \langle 0|H|0\rangle = \frac{\int dp\, 2\omega\sigma^2(p) - 24\lambda_B\pi[1+(15/2\pi)\ln(\overline{\Omega}/m_B)][\int dp\,\sigma(p)/4\pi\omega]^2}{\int dp\,\sigma^2(p)}. \tag{8}$$

We now vary m_2 with respect to $\sigma(p)$ and obtain the integral equation:

$$\sigma(p)\omega(\overline{\Omega},p)[2\omega(\overline{\Omega},p)-m_2] = 6\lambda_B\left[1+\frac{15}{2\pi}\ln\left(\frac{\overline{\Omega}}{m_B}\right)\right]\int\frac{dp\,\sigma(p)}{4\pi\omega(\overline{\Omega},p)}. \tag{9}$$

Since the integral on the right-hand side is a constant, the bound-state wave function $\sigma(p)$ can be read from Eq. (9) directly. The equation can now be integrated to yield an eigenvalue equation for the bound-state mass m_2:

$$\frac{m_R^2}{\lambda_B} = \frac{3}{R\pi}\left[1+\frac{15}{2\pi}\ln\left(\frac{m_R}{m_B}\right)\right]\left(\frac{2\tan^{-1}[(1+R/2)/(1-R/2)]^{1/2}}{[1-(R/2)^2]^{1/2}} - \frac{\pi}{2}\right), \tag{10}$$

where $R = m_2/m_R$, which is our main result.

For comparison we have plotted this equation alongside the perturbative[2] and numerical[3] calculations (Fig. 1). Note that expanding our result for small λ/m_R^2 yields the perturbative expression[2]

$$R = 2\left[1 - \frac{9}{8}\left(\frac{\lambda}{m_R^2}\right)^2 + O(\lambda^3)\right]. \tag{11}$$

This may seem surprising as the perturbative calculation was carried out in a different model, a model normal ordered at m_R and not m_B. However, we find that for $\lambda/m_R^2 < \pi/2$, a region which includes the domain where perturbation theory is valid, the variational calculation yields identical results whether one normal orders at m_R or m_B. As one can see from the figure, our result when expanded to next order provides substantial corrections to the second-order expression. We also note on the topic of comparisons with perturbation theory that in $2+1$ dimensions, the variational method gives for non-normal-ordered $-g_B\phi^4 + \lambda_B\phi^6$,

$$R = 2\left[1 - \exp\left(\frac{-2\pi}{3}R\frac{m_R}{g_R}\right)\right] \tag{12}$$

which possesses nonanalytic behavior in g_R and thus would not appear in any order of perturbation theory.[7]

In the moderate-coupling regime our results are qualitatively similar to the numerical calculations.[3] Both exhibit a minimum value of R at some value of λ/m_R^2. We find the critical value of $\lambda/m_R^2 = \pi/3$, precisely where the alternative solution to the gap equation becomes operative as discussed earlier. We believe that the cusp is an artifact of our Gaussian *Ansatz*. Both the numerical and the variational curves increase beyond this critical value. However, when λ/m_R^2 approaches $\frac{5}{2}[\exp(4\pi/15)-1]$ the variational curve approaches $R=2$ smoothly (with zero first derivative). Beyond this point the stationary condition Eq. (9) is no longer valid. The two-particle energy is now minimized at the variational end point $\sigma(p) = \delta(p)$, i.e., a free two-particle wave function. The numerical calculation continues to rise, $R \to \infty$ as $\lambda/m_R^2 \to \infty$, which is presumably an artifact of the small lattice used.

In conclusion, we confirm the existence of a bound state in $:\lambda(\phi^6 - \phi^4)_2:$. For weak coupling our results agree with perturbation theory and the numerical calculations. At moderate coupling the two-particle binding energy reaches a maximum as indicated by numerical calculations. Finally, at large coupling we find that the binding energy is identically zero corresponding to free-field behavior.

We believe that our results are qualitatively correct given our simple variational *Ansatz*. Of course more accurate variational and/or numerical calculations are needed to determine the quantitative details. Work in this direction is in progress.

The authors gratefully acknowledge conversations with T. Barnes and P. M. Stevenson. The National Science and Engineering Research Council of Canada is thanked for financial assistance.

---

[a]Permanent address: J. W. v. Goethe Universität,

Frankfurt, Germany.

[1]J. Glimm, A. Jaffe, and T. Spencer, in *Constructive Quantum Field Theory,* edited by G. Velo and A. Wightman, Proceedings of the 1973 "Ettore Majorana" International School of Mathematical Physics (Springer-Verlag, Berlin, 1973), p. 165–177.

[2]J. Dimock and J. P. Eckmann, Commun. Math. Phys. **51**, 41 (1976).

[3]T. Barnes and G. J. Daniell, Phys. Lett. **142B**, 188 (1984).

[4]L. I. Schiff, Phys. Rev. **130**, 458 (1963).

[5]P. M. Stevenson, Phys. Rev. D **30**, 1712 (1984); P. M. Stevenson, unpublished.

[6]T. Barnes and G. I. Ghandour, Phys. Rev. D **22**, 924 (1980).

[7]P. M. Stevenson has found the same nonanalytic behavior in g_R for the binding energy (private communication).