# Supersymmetric Quantum Mechanics and Large-$N$ Expansions

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Supersymmetric quantum mechanics is formulated for spherically symmetric potentials in $N$ spatial dimensions. It is seen that the supersymmetric partner potential of a given potential can be effectively treated as being in $N+2$ dimensions. This fact is exploited in calculations using the shifted $1/N$ expansion. Also, the violation of the no-degeneracy theorem in one dimension by the Coulomb potential is seen as a consequence of this result.

Given any one-dimensional potential, supersymmetric quantum mechanics provides a simple recipe for generating a partner potential with the same energy eigenvalues (except for the ground state).[1,2] Often, for many physical problems, it is profitable to deal with partner potentials. This possibility has been used by several people for finding classes of analytically solvable potentials,[3] evaluating the eigenvalues of a bistable potential,[4] studying atomic systems,[5] and improving the WKB approximation.[6]

Recently, large-$N$ expansions,[7-10] particularly those involving shifted expansion parameters,[8-10] have proved to be very useful in the calculation of accurate eigenenergies for spherically symmetric potentials. The purpose of this paper is to show that the use of supersymmetric partner potentials can be exploited to improve further both the accuracy and the simplicity of large-$N$ expansions. Typically, we find that just the leading term in a shifted large-$N$ expansion using partner potentials yields all the energy levels correctly to three significant digits for essentially any three-dimensional spherically symmetric potential of physical interest. Further accuracy is easily obtained from previously calculated higher-order terms.[9]

We first review the ideas of supersymmetric quantum mechanics and set up the formalism for spherically symmetric potentials in $N$ spatial dimensions. This will demonstrate that the supersymmetric partner of a given potential can be effectively treated as being in $N+2$ spatial dimensions.[11] This fact is responsible for substantially improving the convergence of large-$N$ expansions. Several useful, illustrative potentials (Hulthén logarithmically screened Coulomb, quarkonium) are treated in order to demonstrate our approach. The Coulomb potential in one dimension is somewhat special since it is its own supersymmetric partner; the curious consequences of this (like intersecting energy levels and degeneracy in one dimension[12]) are discussed.

Assume that one has a potential $V^0(x)$ whose ground-state wave function $\psi_0(x)$ is known, and whose ground-state energy has been adjusted so that $E_0 = 0$. Then the Schrödinger equation for the ground state is ($\hbar = m = 1$)

$$H^0 \psi_0 \equiv \left[ -\frac{1}{2} \frac{d^2}{dx^2} + V^0(x) \right] \psi_0 = 0, \tag{1}$$

and consequently,

$$H^0 = \frac{1}{2} \left[ -\frac{d^2}{dx^2} + \frac{\psi_0''}{\psi_0} \right]. \tag{2}$$

Define the operators

$$Q^{\pm} \equiv \frac{1}{\sqrt{2}} \left[ \mp \frac{d}{dx} - \frac{\psi_0'}{\psi_0} \right]. \tag{3}$$

This gives

$$Q^+ Q^- = H^0,$$

$$Q^- Q^+ \equiv H^1 = -\frac{1}{2} \frac{d^2}{dx^2} + V^1(x), \tag{4}$$

where

$$V^1(x) = V^0(x) - \frac{d}{dx} \left( \frac{\psi_0'}{\psi_0} \right) = -V^0(x) + \left( \frac{\psi_0'}{\psi_0} \right)^2. \tag{5}$$

If $\psi_n$ is any eigenfunction of $H^0$ with eigenvalue $E_n$, then $Q^- \psi_n$ is an eigenfunction of $H^1$ with the same eigenvalue $E_n$. $V^1$ is the supersymmetric partner of $V^0$. It has the same energy levels as $V^0$ (except for $E_0 = 0$).

When making a large-$N$ expansion for an arbitrary spherically symmetric potential $V(r)$, the effective potential appearing in the radial Schrödinger equation is

$$V^0(r) = \frac{(k-1)(k-3)}{8r^2} + V(r), \tag{6}$$

where $k \equiv N + 2l$, $l$ being the $N$-dimensional orbital angular-momentum quantum number. Eigenstates are labeled by $k$ and the radial quantum number $n = 0, 1, 2, \ldots$ . As $r \to 0$, the $n = 0$ wave function behaves like $\psi_0(r) \sim r^{(k-1)/2}$. If one sets

$$\psi_0(r) = r^{(k-1)/2} \Phi_0(r), \tag{7}$$

where $\Phi_0(r)$ is finite at the origin, then Eq. (5) immediately gives the supersymmetric partner of $V^0(r)$ to be

$$V^1(r) = \frac{(k+1)(k-1)}{8r^2} + V(r) - \frac{d}{dr}\left[\frac{\Phi_0'}{\Phi_0}\right]. \quad (8)$$

$V^1(r)$ and $V^0(r)$ have the same energy eigenvalues, but large-$N$ expansions with $V^1(r)$ are far better since the angular-momentum barrier in Eq. (8) is $(k' - 1)(k' - 3)/8r^2$, where $k' = k + 2$, i.e., effectively one is working in two extra spatial dimensions! (Equivalently, $l$ has increased by one.[5]) Thus, for example, in order to calculate the energy of the state with quantum numbers $k, n$ of $V^0(r)$, one can equally well use $k' = k + 2$, $n - 1$ with $V^1(r)$. We will now demonstrate this procedure with some explicit examples. These examples are conveniently classified into two categories depending on whether the lowest-energy $(n = 0)$ wave function for a given $k$ is analytically known or not.

In atomic physics, the Hulthén potential is widely used. For $N = 3$, $l = 0$, the effective potential is[13]

$$V_H^0(r) = -\frac{\delta e^{-\delta r}}{1 - e^{-\delta r}} + \frac{(2-\delta)^2}{8}; \quad (9)$$

$$\psi_0(r) \propto (1 - e^{-\delta r})e^{-(2-\delta)r/2} \quad (\delta < 2).$$

With use of Eq. (5), the supersymmetric partner of $V_H^0$ is

$$V_H^1 = V_H^0 + \frac{\delta^2 e^{-\delta r}}{(1 - e^{-\delta r})^2}. \quad (10)$$

As $r \to 0$, $V_H^1(r) \sim 1/r^2$, which, as mentioned before, corresponds to the angular-momentum barrier $(k' - 1)(k' - 3)/8r^2$ for $k' = 5$ $(N = 5, l = 0)$. Table I shows the results for the energies of the $2s$ state of the Hulthén potential $V_H^0$ with $\delta = 0.05$ in three dimensions $(n = 1, l = 0)$ and the same energies obtained from the partner potential $V_H^1$ which has $N = 5$, using the shifted large-$N$ expansion formulas of Ref. 9. Also shown in Table I are the results for the "logarithmically screened Coulomb (LSC) potential" $(\delta = 0.05,$

$N = 3, l = 0)$

$$V_L^0(r) = -\frac{(2-\delta)\delta^2 r - 2\delta}{2(1+\delta r)^2 \ln(1+\delta r)} + \frac{(2-\delta)^2}{8};$$

$$\psi_0(r) \propto \ln(1+\delta r)e^{-(2-\delta)r/2} \quad (\delta < 2). \quad (11)$$

(This potential was chosen since its ground state has a simple analytic form. It has many possible future applications in problems involving modified Coulomb potentials. Its behavior is qualitatively similar to both the Hulthén and the Yukawa potentials; see Fig. 1.)

If the lowest-energy wave function for given $N$ and $l$ is not analytically known, one can, of course, determine it via numerical integration of the Schrödinger equation. This immediately yields the partner potential $V^1(r)$, which can be used in large-$N$ calculations for all values of the radial quantum number $n$. As an example we have determined the first excited-state energy of the quarkonium-type potential,[15]

$$V_Q^0(r) = \frac{1}{2}\left[\frac{r}{(2.34)^2} - \frac{0.52}{r}\right] - E_0,$$

$$E_0 = 0.239\,06, \quad (12)$$

for $N = 3$, $l = 0$. The result is shown in Table I.

It is clear from Table I that, although excellent results are obtained with use of the shifted $1/N$ expansion for the original potential $V^0(r)$ in three dimensions, even faster convergence results from use of the supersymmetric partner potential in five dimensions. Thus, for many applications considerable analytic simplification occurs since it is sufficient to use just the leading term in the shifted $1/N$ expansion for $V^1(r)$.

There is another virtue in using $V^1(r)$. If we are dealing with a bistable potential $V^0(r)$, the two lowest-energy levels are often closely spaced. However, its supersymmetric partner $V^1(r)$ does not have the lowest-energy state and oscillates less than $V^0(r)$ (it is essentially only a single well).[4] For this reason alone a shifted large-$N$ expansion of $V^1(r)$ (which is an expansion about one stable minimum) would be preferable.

TABLE I. Energies for the first excited state of various potentials $(\hbar = m = 1)$. $E^{(0)}$, $E^{(1)}$, and $E^{(2)}$ represent the results from the shifted large-$N$ expansion (Ref. 9) keeping just the leading, two, and three terms, respectively.

| Potential | Exact energy[a] | Results with $N = 3$ using $V^0(r)$ $(n = 1, l = 0)$ | | | Results with $V^1(r)$ $(N = 5)$ $(n = 0, l = 0)$ | | |
|---|---|---|---|---|---|---|---|
| | | $E^{(0)}$ | $E^{(1)}$ | $E^{(2)}$ | $E^{(0)}$ | $E^{(1)}$ | $E^{(2)}$ |
| Hulthén [Eq. (9)][b] | −0.101 25 | −0.101 456 | −0.101 249 | −0.101 250 | −0.101 250 | −0.101 250 | −0.101 250 |
| LSC [Eq. (11)][c] | −0.103 96 | −0.104 65 | −0.103 97 | −0.103 96 | −0.104 00 | −0.103 96 | −0.103 96 |
| Quarkonium [Eq. (12)][d] | 0.568 | 0.573 | 0.574 | 0.573 | 0.569 | · · · | · · · |

[a] Reference 14.

[b] $V_H^0(r) + E_H^0$; $E_H^0 = -\frac{1}{8}(2-\delta)^2$, $\delta = 0.05$.

[c] $V_L^0(r) + E_L^0$; $E_L^0 = -\frac{1}{8}(2-\delta)^2$, $\delta = 0.05$.

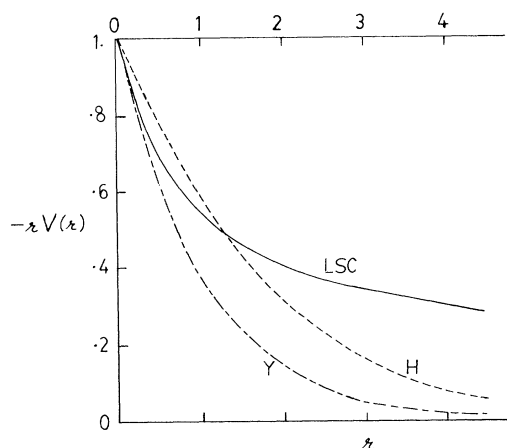[d] $V_Q^0(r) + E_Q^0$; $E_Q^0 = 0.239\,06$.

FIG. 1. Plot of $-rV(r)$ vs $r$ with $V(r)$ given by the Hulthén potential (H), Eq. (9), the logarithmically screened Coulomb potential (LSC), Eq. (11), and the Yukawa potential (Y), $V(r) = -e^{-\delta r}/r$. For purposes of comparison all potentials have been chosen to vanish as $r \rightarrow \infty$ and the parameter $\delta = 1$. The unscreened Coulomb potential corresponds to the line $-rV(r) = 1$.

FIG. 2. Low-lying energy levels of the one-dimensional power-law potentials $V(x) = -|x|^{\nu}$ vs $\nu$ for $-1.25 < \nu < 0$, as given by the shifted $1/N$ expansion (Ref. 9). Curves are labeled by the number of nodes in the wave function of the given states.

Finally, we consider the Coulomb potential in one dimension. For any symmetric one-dimensional potential $V(x)$, the effective potential of Eq. (6) with $k = 1$ gives only even eigenstates since $\psi_0(x) \sim$ const as $x \rightarrow 0$ [Eq. (7)]. The partner potential will have $k' = 3$, corresponding to states in three dimensions or odd states in one dimension since $\psi_0(x) \sim x$ as $x \rightarrow 0$. The Coulomb potential $V(x) = -A/|x|$ is special since $V^0(x)$ with $k = 1$ is identical to its partner $V^1(x)$ with $k' = 3$. That is, the odd states of the one-dimensional Coulomb potential are degenerate with the even states of the same potential (except for the ground state).[12] This is perhaps the only potential that violates the no-degeneracy theorem in one dimension,[12] and this can here be seen to result from supersymmetry. The behavior of the energy levels and wave functions of one-dimensional power-law potentials $V(x) = A|x|^{\nu}$ near $\nu = -1$ is rather curious and can be studied reliably in the shifted $1/N$ expansion since its results are exact at $\nu = -1$.[8-10] We find that the odd and even energy levels cross each other at $\nu = -1$ (see Fig. 2). As $\nu$ decreases through $-1$, the ground state (whose energy is $-\infty$ at $\nu = -1$) disappears, and each of the remaining even states loses two nodes in their wave function as their energy falls below an odd state.[16] The variation in the number of nodes for each state as a function of $\nu$ is shown in Fig. 2. Note that for any given $\nu$ the number of nodes increases with energy, and the ground-state wave function is nodeless, as expected.

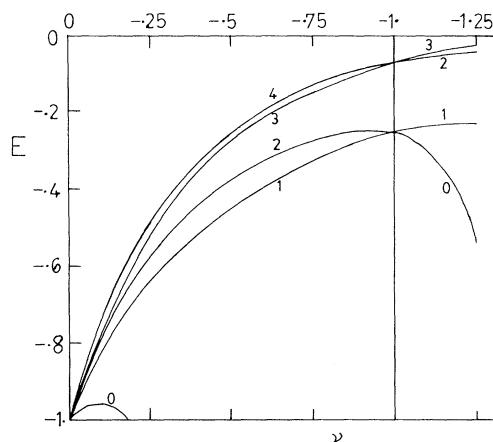We thank W. Y. Keung, J. Maharana, and A. Pagna-

[1]E. Witten, Nucl. Phys. **B185**, 513 (1981); F. Cooper and B. Freedman, Ann. Phys. (N.Y.) **146**, 262 (1983).

[2]E. Schrödinger, Proc. Roy. Irish Acad. **A46**, 9 (1940), and **A46**, 183 (1941); L. Infeld and T. E. Hull, Rev. Mod. Phys. **23**, 21 (1951); A. Andrianov, N. Borisov, and M. Ioffe, Pis'ma Zh. Eksp. Teor. Fiz. **39**, 78 (1984) [JETP Lett. **39**, 93 (1984)]; P. Deift, Duke Math. J. **45**, 267 (1978).

[3]L. Gendenshtein, Pis'ma Zh. Eksp. Teor. Fiz. **38**, 299 (1983) [JETP Lett. **38**, 356 (1983)].

[4]M. Bernstein and L. S. Brown, Phys. Rev. Lett. **52**, 1933 (1984).

[5]V. Kostelecký and M. M. Nieto, Phys. Rev. Lett. **53**, 2285 (1984); V. Kostelecký, M. M. Nieto, and D. Truax, Los Alamos National Laboratory Report No. LA-UR-84-3972, 1984 (to be published).

[6]A. Comtet, A. Bandrauk, and D. Campbell, Phys. Lett. **105B**, 159 (1985).

[7]L. Mlodinow, in *Quarks and the Nucleus*, Progress in Particle and Nuclear Physics, Vol. 8, edited by D. Wilkinson (Pergamon, New York, 1982), and thesis, University of California, Berkeley, California, 1981 (unpublished); L. Mlodinow and N. Papanicolaou, Ann. Phys. (N.Y.) **128**, 314 (1980), and **131**, 1 (1981); E. Witten, in *Recent Developments in Gauge Theories*, edited by G. 't Hooft *et al.* (Plenum, New York, 1980); C. Bender, L. Mlodinow, and N. Papanicolaou, Phys. Rev. A **25**, 1305 (1982); S. Kalara, University of Rochester Report No. UR-812, 1982 (unpublished); L. Mlodinow and M. Shatz, J. Math. Phys. **25**, 943 (1984); J. Ader, Phys. Lett. **97A**, 178 (1983); R. Gango-

padhyay *et al.*, Phys. Rev. A **30**, 594 (1984); L. Yaffe, Rev. Mod. Phys. **54**, 407 (1982); M. Sinha-Roy *et al.*, J. Phys. A **17**, L687 (1984); J. Miramontes and C. Pajares, Nuovo Cimento **84**, 10 (1984); S. Hikami and E. Brezin, J. Phys. A **12**, 759 (1979).

[8]U. Sukhatme and T. Imbo, Phys. Rev. D **28**, 418 (1983); T. Imbo, A. Pagnamenta, and U. Sukhatme, Phys. Lett. **105A**, 183 (1984); B. Lauer, University of Illinois at Chicago Report No. UIC-85-11, 1985 (unpublished); A. Flitney, unpublished. For a review of shifted large-$N$ expansions, see U. Sukhatme, in *Proceedings of the Seventh Warsaw Symposium on Elementary Particle Physics, Kazimierz, Poland, 1984,* edited by Z. Ajduk (Institute of Theoretical Physics, Warsaw University, Warsaw, 1984).

[9]T. Imbo, A. Pagnamenta, and U. Sukhatme, Phys. Rev. D **29**, 1669 (1984).

[10]T. Imbo and U. Sukhatme, Phys. Rev. D **31**, 2655 (1985).

[11]This is reminiscent of the work of G. Parisi and N. Sourlas, Phys. Rev. Lett. **43**, 744 (1979), and **46**, 871 (1981), where a decrease in the number of spatial dimensions by two was exploited for calculation of the properties of a magnetic system in a random magnetic field. We thank L. Carson for this comment.

[12]R. Loudon, Am. J. Phys. **27**, 649 (1959); M. M. Nieto, Am. J. Phys. **47**, 1067 (1979).

[13]C. Lam and Y. Varshni, Phys. Rev. A **4**, 1875 (1971).

[14]Numerical integration was performed with use of the standard Runge-Kutta method for ordinary differential equations. The program used was typically accurate to six significant figures.

[15]E. Eichten *et al.*, Phys. Rev. D **21**, 203 (1980).

[16]This node reduction can be seen in the improved shifted-$(1/N)$-expansion wave functions presented in Ref. 10.