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Statistical Mechanical Origin of the Entropy of a Rotating, Charged Black Hole

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It is shown that the entropy of a rotating, charged black hole is, in senses made precise in the paper, (i) the logarithm of the number of quantum mechanically distinct ways that the hole could have been made, and (ii) the logarithm of the number of configurations that the hole's "atmosphere," as measured by stationary observers, could assume in the presence of its background noise of acceleration radiation. In addition, a proof is given of the generalized second law of thermodynamics.

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Hawking[1] has shown that a stationary black hole emits its thermal radiation precisely as though it were endowed with a temperature $T_H = \hbar g_H/2\pi$, where g_H is its classically defined surface gravity and we use units with $G=c=k_B=1$. By comparing this result with the classical laws of black-hole mechanics,[2] Hawking has also inferred[1,3] that a black hole must be endowed with an entropy $S_H = A_H/4\hbar$, where A_H is the classically defined area of its horizon.

This value for the entropy is in accord with a previous order-of-magnitude inference by Bekenstein.[4] Bekenstein has also conjectured[4,5] that $W \equiv e^{S_H}$ must in some sense be the number of "quantum mechanically distinct internal states" that a black hole could have, corresponding to its classically observed external parameters. Despite strong indirect evidence for Bekenstein's conjecture,[6] the physical nature of a hole's "internal states" has remained a puzzle[7]; and consequently there has been no satisfactory statistical mechanical derivation of the entropy S_H. Three answers to this puzzle have been proposed: (i) Gerlach's[8] view of Hawking radiation as produced by zero-point fluctuations on the surface of the star that collapsed to form the hole, and his conclusion that the number W_{zp} of zero-point fluctuation modes that give rise to the Hawking radiation of a freely evaporating Schwarzschild hole satisfies $\ln W_{zp} \cong 280 S_H$. (ii) York's[9] view of Hawking radiation as produced by the hole's "quantum ergosphere" of thermally excited gravitational quasinormal modes, and his conclusion that the number of ways W_{qe} that this quantum ergosphere can be excited and reexcited, during the evaporation of a Schwarzschild hole into a surrounding radiation bath, satisfies $\ln W_{qe} \cong 1.106\,17 S_H$. (iii) A view implicit in the writings of Bekenstein[4–6] and Hawking[3] that $W \equiv e^{S_H}$ might be the number of quantum mechanically distinct ways that the black hole could have been made by infalling quanta (particles). In this paper we shall pursue this third view and from it shall obtain a precise statistical mechanical explanation of W.

We begin with an order-of-magnitude derivation of the number of ways that a Schwarzschild black hole of mass M can be made by accretion of quanta from "infinity" (i.e., $r \gg 2M$). We shall insist that the hole be made in a total time less than the *Hawking evaporation time*[1] $t_H \cong 2560\pi \nu_\phi^{-1} M^3/\hbar = 640 \nu_\phi^{-1} N_B M$, so that it does not evaporate during construction. Here and below "$\cong$" means "is equal to, aside from factors of order unity;" ν_ϕ is the effective number of distinct kinds of field modes (i.e., distinct particle species, helicities, etc.) into which the hole can radiate[10]; and $N_B \equiv 4\pi M^2/\hbar$ is the hole's dimensionless *Bekenstein number,* which is also equal to the nonrotating hole's entropy S_H.

The vast majority of ways to make the hole involve building it up, bit by bit, by the accretion of one quantum after another, with the individual quantum energies ϵ kept as small as possible: $\epsilon \cong \hbar/4M$, corresponding to a wavelength of order the hole's Schwarzschild radius.[4,11] The total number of such quanta required to make the hole is $n_\infty \cong M/\epsilon \cong N_B$. The total number of single-particle quantum states in which these n_∞ quanta can be injected, N_∞, is given by the total phase-space volume that they occupy before injection, $\simeq 27\pi\epsilon^3 M^2 t_H$, multiplied by the density of single-particle states in phase space, $\nu_\phi/(2\pi\hbar)^3$, i.e., $N_\infty \cong (135/4\pi^2)N_B \cong N_B$. The number of ways to make the black hole, W_∞, is then the number of ways to distribute the $n_\infty \cong N_B$ quanta (assumed, for simplicity, to be bosons) among the $N_\infty \cong N_B$ states,

$$W \cong (N-1+n)!/[(N-1)!n!], \tag{1}$$

and the logarithm of this in good accord with Bekenstein's conjecture,

$$\ln W_\infty \cong N_B = S_H. \tag{2}$$

This result is not surprising, since our accreting particle configurations are an approximate "time reversal" of the products of black-hole evaporation, which are known to have a statistical mechanical entropy $S = \ln W_\infty \simeq \frac{4}{3}N_B$.[12]

The modes used in the above analysis are all accessible from outside the hole's potential barrier, $r > 4M$. However, these "outsider" modes are far outnumbered by the "insider" modes accessible to a "demon" at a proper distance $z = 4M(1-2M/r)^{1/2} << 2M$ above the horizon. The number of insider modes is

$$N_z = (4\pi/\theta_{esc}^2)N_\infty \cong (M/z)^2 N_\infty \cong (M/z)^2 N_B,$$

where $\theta_{esc} = (3^{3/2}/8)z/M$ is the "escape angle" into which the momenta of infalling outsider modes are focused as they fly past the demon [Box 25.7 of Misner, Torne, and Wheeler (MTW)[13]]. Because the insider modes at height z must have vertical wavelengths $\lesssim z$, their red-shifted energies (the amount by which one of their quanta can change the hole's mass) are $\Delta M \gtrsim (\hbar/z)(1-2M/r)^{1/2} \cong \hbar/4M = \epsilon$; and thus the maximum number of quanta that can be injected into them to make the hole is $n_z \cong M/\epsilon \cong N_B$.

If we do not invoke some sort of cutoff, the number of insider modes will be infinite ($\lim_{z\to 0} N_z = \infty$), and correspondingly the number of ways to make the black hole using them will be infinite. One conceivable cutoff, the Planck length, leads to $N_z \cong N_B^2$ and thence, via Eq. (1) with $n_z \cong N_B << N_z \cong N_B^2$, to

$$\ln W_z \cong n_z \ln(N_z/n_z) \cong N_B \ln N_B, \tag{3}$$

in serious violation of Bekenstein's conjecture. Even for the much more conservative cutoff of $r = 2M + \hbar^{1/2}$, corresponding to $z = (8M\hbar^{1/2})^{1/2}$, one obtains $N_z = N_B^{3/2}$, which leads again to Eq. (3).

Fortunately, this insider calculation is flawed by our failure to take account of the thermal "acceleration radiation," which quantum field theory predicts[14,15] to be measured by particle detectors (and demons) at rest near a black-hole horizon. To see quickly how this removes the $\ln N_B$ from Eq. (3), consider the related issue of the number of ways to inject one quantum of energy E into $N >> 1$ identical states. If the N states are initially empty (analog of our above, incorrect, insider calculation), then the number of ways is Eq. (1) with $n = 1$, which gives $\ln W = \ln N$ [analog of our incorrect $\ln N_B$ factor in Eq. (3)]. By contrast, if the N states are initially thermalized at a temperature T, then they contain already $n = N/(e^{E/T}-1)$ quanta distributed randomly, and the number of distinct ways to inject one more quantum is

$$W = \frac{[\text{Eq. (1) with } n \text{ replaced by } n+1]}{[\text{Eq. (1)}]} = \frac{(N+n)}{n} = e^{E/T},$$

so that

$$\ln W = \frac{E}{T} = \frac{(\text{energy injected})}{T} = (\text{change in entropy}). \tag{4}$$

Thus, the thermal population of the states converts the offending $\ln N$ into the desired thermodynamic relation.

With this as motivation, we shall now compute precisely the number of ways to make a slowly evolving, charged, rotating, axisymmetric black hole by injecting quanta into its thermalized "atmosphere." Our computation will be based on observables measured by a family of fiducial observers (referred to as FIDOS elsewhere[16]; analogs of the demon used above) who, very near the hole, are at rest with respect to the horizon's generators. In the limit that the hole's evolution is ignored, these fiducial observers are the zero-angular-momentum observers (ZAMO's) of Bardeen.[17] In the evolving case their world lines snuggle up to the horizon as shown in Fig. 1. These fiducial observers see as simultaneous all events on a spacelike hypersurface $\mathcal{S}_t$ of constant "universal time" t, which is orthogonal to their world lines and which, consequently, is asymptotic to the horizon $\mathcal{H}$ as shown in

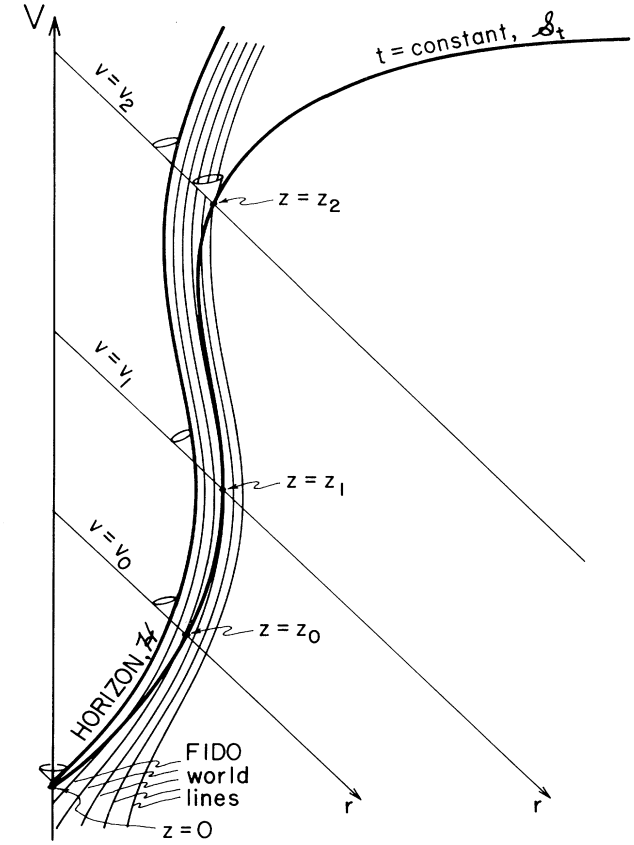

FIG. 1. Space-time diagram, with an ingoing null time coordinate v plotted vertically and a radial coordinate r plotted down and to the right, for the evolution of a black hole. The horizon $\mathscr{H}$ initially grows due to accretion, then shrinks due to evaporation, and then grows again.

Fig. 1. Because of this asymptotic behavior, the fiducial observers see layered into $\mathscr{S}_t$, between proper distances z_0 and z_1 above the horizon, the entire history of the horizon's evolution between horizon times v_0 and v_1 (Fig. 1).[18] In particular, the hole's mass M, angular momentum J, charge Q, angular velocity Ω_H, surface gravity g_H, temperature T_H, surface area A_H, entropy S_H, and electrical potential Φ_H, which are normally regarded as functions of time v on the horizon $\mathscr{H}$, can equally well be regarded as functions of height z in $\mathscr{S}_t$.

The evolution of the horizon $\mathscr{H}$ is normally expressed in terms of integrals over $\mathscr{H}$ of the expectation values $\langle T^{\mu\nu}\rangle$ and $\langle J^{\mu}\rangle$ of the stress-energy tensor (including gravitons) and the charge-current four-vector.[19] When translated into laws for the fiducial-observer–measured vertical structure in $\mathscr{S}_t$, the horizon evolution laws become equations for $\Delta M \equiv M(z_1)-M(z_0)$, $\Delta J \equiv J(z_1)-J(z_0)$, and $\Delta Q \equiv Q(z_1)-Q(z_0)$:

$$\Delta M = \int_{z_0}^{z_1} (\langle -T_{\hat 0}^{\hat 0}\rangle - \langle J^{\hat 0}\rangle A_0) A_H\, dz, \tag{5a}$$

$$\Delta J = \int_{z_0}^{z_1} (\langle T_{\hat\phi}^{\hat 0}\rangle + \langle J^{\hat 0}\rangle A_\phi) A_H\, dz, \tag{5b}$$

$$\Delta Q = \int_{z_0}^{z_1} \langle J^{\hat 0}\rangle A_H\, dz. \tag{5c}$$

Here ϕ, 0, and $\hat 0$ denote components along the (almost) Killing vector ξ_ϕ which generates rotations about the axis of symmetry, along the (almost) Killing vector ξ_0 which far from the hole and at $v_0 < v < v_1$ generates time translations, and along the fiducial-observer four-velocity $\mathbf{U} = \alpha^{-1}(\xi_0 + \Omega_H \xi_\phi)$ (with α a z-dependent gravitational red-shift factor); A_μ is the hole's electromagnetic four-potential.

The fundamental fiducial-observer–measured observables are not $\langle T^{\mu\nu}\rangle$ and $\langle J^{\mu}\rangle$, but rather the number of quanta in the fiducial-observer–defined single-particle states $|I\rangle$ of the quantum fields (including gravitons) which contribute to $T^{\mu\nu}$ and J^{μ}. The states $|I\rangle$ are eigenstates of charge q and of the constants of the motion $\epsilon \equiv -\pi_0 = -(p_0 + qA_0)$ and $j \equiv \pi_\phi = p_\phi + qA_\phi$ for classical charged particles (MTW,[13] Secs. 33.5–33.8). [Here π_μ is a particle's canonical momentum, p_μ its kinetic momentum, and ϵ and j its conserved "energy-at-infinity" (red-shifted energy) and "angular momentum."] We shall denote by q_I, ϵ_I, and j_I the eigenvalues of q, ϵ, and j in state $|I\rangle$, and shall denote $-p_{0I} \equiv \epsilon_I + qA_0$, $p_{\phi I} \equiv j_I - q_I A_\phi$.

It is a particle's kinetic energy-at-infinity $-p_0$ and kinetic angular momentum p_ϕ that contribute to $\langle -T_{\hat 0}^{\hat 0}\rangle$ and $\langle T_{\hat\phi}^{\hat 0}\rangle$. However, $\langle T_{\hat\mu}^{\hat 0}\rangle$ is not $\sum n_I p_{\mu I}/V$ with n_I the mean occupation number of state $|I\rangle$ in volume V, as it would be if the fiducial observers were inertial observers in flat space-time. Rather, $\langle T_{\hat\mu}^{\hat 0}\rangle$ is produced only by the *excess* particles over those in a thermal bath of acceleration radiation,[20] which has mean occupation number $\bar n_I = (e^{E_I/T} \pm 1)^{-1}$ (+ for fermions, − for bosons). Here E_I is the fiducial-observer–measured energy of a particle in state $|I\rangle$, $T \equiv \hbar a/2\pi = T_H/\alpha$ is the fiducial-observer–measured bath temperature, and $a = g_H/\alpha$ is the fiducial observer's acceleration, so that

$$E_I \equiv -\mathbf{U}\cdot\mathbf{p}_I = \alpha^{-1}(-p_{0I} - \Omega_H p_{\phi I}) = \alpha^{-1}(\epsilon_I - \Omega_H j_I - \Phi_H q_I),$$

where $\Phi_H \equiv -A_0 - \Omega_H A_\phi$ is the hole's electric potential. Thus,

$$\bar n_I = [\exp(\epsilon_I - \Omega_H j_I - \Phi_H q_I)/T_H \pm 1]^{-1}, \tag{6}$$

and the integrals of $\langle T_{\hat\mu}^{\hat 0}\rangle$ and $\langle J^{\hat 0}\rangle$ over the three-volume $z_0 < z < z_1$ on $\mathscr{S}_t$ are[20]

$$\int \langle T^{\hat{0}}_{\mu} \rangle A_H \, dz = \sum (n_I - \bar{n}_I) p_{\mu I}, \quad (7a)$$

$$\int \langle J^{\hat{0}} \rangle A_H \, dz = \sum (n_I - \bar{n}_I) q_I. \quad (7b)$$

Here the sum is over all states $|I\rangle$ which are spatially confined to $z_0 < z < z_1$.

By combining Eqs. (5) and (7) we obtain

$$\Delta M = \sum (n_I - \bar{n}_I)\epsilon_I, \quad \Delta J = \sum (n_I - \bar{n}_I) j_I, \quad \Delta Q = \sum (n_I - \bar{n}_I) q_I. \quad (8)$$

These equations say that the fiducial observers measure the hole's change in mass ΔM, angular momentum ΔJ, and charge ΔQ, between heights z_0 and z_1, to be equal to the *excess* energy-at-infinity ϵ, angular momentum j, and charge q over the ϵ, j, and q of the fiducial-observer–measured acceleration-radiation bath.

Although the bath contributes nothing to ΔM, ΔJ, and ΔQ, individual bath quanta are physically indistinguishable by fiducial-observer measurements from quanta injected by an external physicist or internal demon.[15] Consequently, if we ask how many quantum mechanically distinct ways there are to generate the ΔM, ΔJ, and ΔQ of Eqs. (8) by injecting quanta into the states $|I\rangle$, the answer will be the same as in the standard problem of injecting new quanta into a thermal bath with temperature T_H and thermodynamic potentials Ω_H and Φ_H [Eq. (6)]:

$$W_{z_0 \to z_1} = \exp[(\Delta M - \Omega_H \Delta J - \Phi_H \Delta Q)/T_H] = \exp(\Delta S_H). \quad (9)$$

Here ΔS_H is the change in black-hole entropy between z_0 and z_1 as defined thermodynamically by Hawking.[1,3] This is *precisely* the answer we were hoping for!

Actually, Eq. (9) is the number of ways to do the injection only if ΔM, ΔJ, and ΔQ are small enough to not disturb the thermal bath significantly; i.e., only if $\Delta M << \sum \bar{n}_I \epsilon_I \sim (M/z_0)^2 (\hbar/M)$, which means $z_0 << \hbar^{1/2}(M/\Delta M)^{1/2}$, and similarly for ΔJ and ΔQ. Now, $z_0/M \sim \exp(-g_H \Delta t)$, where Δt is the universal time (proper time measured far above the horizon) since the piece of atmosphere at z_0 was laid down. Thus, after the very short time $\Delta t \sim g_H{}^{-1} \ln N_B \sim (4 \times 10^{-3} \text{ sec})(M/M_\odot)$, z_0 is of order $\hbar^{1/2}$ and ΔM has sunk low enough into the thermal atmosphere to not disturb it significantly.

By taking the logarithm of Eq. (9) and summing over all layers in the hole's atmosphere, we see that $S_H = \ln W$, where $S_H \equiv$ (total entropy of the hole as defined by Hawking[2]) and $W \equiv$ (the number of quantum mechanically distinct ways to generate the hole's total M,J,Q by injecting quanta into its thermal atmosphere). Put differently, S_H is the logarithm of the number of distinct configurations which could exist in the hole's thin, layered atmosphere in the presence of its background noise of acceleration radiation. Put yet differently, S_H is the logarithm of the amount of information that one loses when one "stretches the horizon," in the black-hole "membrane formalism,"[20] to cover up its thin atmosphere. From this viewpoint, black-hole entropy is only skin deep.

The above analysis provides, as a side product, a proof of the generalized second law of thermodynamics[3,4,22]—that in any process involving the interaction of a black hole with the external universe, the sum of the hole's entropy and the universe's entropy cannot decrease. The proof: Since the hole's atmosphere plays the role of a thermal bath which exchanges particles with the universe, and since (when one used energy-at-infinity ϵ and Hawking temperature T_H instead of locally measured energy E and temperature T) the change in the hole's entropy is precisely that associated with a standard thermal bath, the generalized second law is merely a special case of the ordinary second law.

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[20]This is actually our own highly plausible conjecture. It has not yet been proved in full generality by quantum field theory calculations, but it is consistent with all calculations to date of $\langle T^{\mu\nu}\rangle$ and of the spectra of fiducial-observer–measured quanta near the horizon of a black hole; see, e.g., Sciama *et al.,* Ref. 19. For example, Eqs. (7) agree with calculations very near a Schwarzschild horizon for the Boulware vacuum where $n_l = 0$ (no particles at all) and $\langle T_{\hat{0}}^{\hat{0}}\rangle$ is the negative of that of a thermal bath, and for the Hartle-Hawking vacuum where $n_l = \bar{n}_l$ (perfect thermal bath) and $\langle T_{\hat{0}}^{\hat{0}}\rangle$ vanishes asymptotically.
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