Simple Realization of the Inflationary Expansion of the Universe

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The time evolution of the N-vector model coupled to the Einstein field equation in a Robertson-Walker metric is studied numerically under the assumption of initial equilibrium at high temperatures and in the limit as $N \to \infty$. The system shows inflationary expansion of the Robertson-Walker scale factor, $a(t)$, over many e-foldings as well as local ordering. The associated structure factor develops a Bragg peak near zero wave number which has the proper weight for a spontaneously broken symmetry and has a width (inversely proportional to a characteristic domain size) which is inversely proportional to $a(t)$.

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The nature of the evolution of the early universe is a topic of enormous complexity and interest.[1] The various proposed "inflationary" scenarios[2–4] are appealing in certain respects from a physical point of view, but as has been recently pointed out,[5] certain aspects of the process, as typically presented, seem unphysical. In particular, the nonequilibrium nature of the problem does not seem to be properly taken into account. Part of the difficulty is that the complete problem, which involves the coupling of some nontrivial quantum field theory[6] to Einstein's equations in a strongly nonequilibrium setting, is simply too difficult to handle. In this paper I treat the more tractable problem of the coupling of Einstein's equation to a field theory which has the desirable features of solvability and a nontrivial phase structure. The field theory is the N-vector model[7] in the $N \to \infty$ limit. I find inflation and ordering as a natural consequence in this model.

The basic model I study assumes a Robertson-Walker metric where the scale factor, $a(t)$, is driven in time t by the average energy density $\langle \rho \rangle$ via the Einstein field equation[8]

$$(\dot{a}/a)^2 = (8\pi/3)\langle \rho \rangle. \tag{1}$$

The energy density is assumed to be given by

$$\rho = N^{-1}[\tfrac{1}{2}\textstyle\sum_i \Pi_i^2 + \tfrac{1}{2}a^{-2}(t)(\nabla\psi_i)^2 + V(\psi)], \tag{2}$$

where $\psi_i(\mathbf{x},t)$ is an N-component field, Π_i is the canonically conjugate momentum, and $V(\psi)$ is the potential assumed to be of the form

$$V(\psi) = \frac{1}{4uN}\left(Nr + u\sum_{i=1}^{N}\psi_i^2\right)^2. \tag{3}$$

The equations of motion[9] satisfied by the fields are

$$\dot{\psi}_i = \Pi_i, \tag{4}$$

$$\dot{\Pi}_i = -3H\Pi - \partial V/\partial\psi_i + a^{-2}\nabla^2\psi_i, \tag{5}$$

where $H = \dot{a}/a$ is Hubble's "constant."

The average of ρ in (1) is over an initial probability distribution. While I will assume that the system is initially in thermal equilibrium at temperature T_0, other initial conditions could easily be studied.[10] The determination of $\langle \rho(t) \rangle$ requires knowledge of the correlation functions between the $\psi_i(\mathbf{x},t)$ and $\Pi_i(\mathbf{x},t)$ which are also averaged over the initial equilibrium state. A second, and less restrictive, assumption is that we have a classical system. This requires, for the usual reason, that I introduce a short-distance cutoff in the theory which restricts wave numbers to be less than a cutoff Λ. One can model the effects of quantum statistics, which naturally provide a large wave-number cutoff, via the Planck distribution by choosing $\Lambda = T_0$. Thus, the average energy density will go as T_0^4 for large T_0 in equilibrium. It does not seem difficult to treat the full quantum-mechanical problem except that the numerical solution appears to be more complicated.

Because of the symmetry in the problem the average values of the fields $\langle \psi_i(\mathbf{x},t) \rangle$, $\langle \Pi_i(\mathbf{x},t) \rangle$, which are initially zero, remain zero.[11] Thus, nontrivial information about the ordering process is contained in the correlation functions[12]

$$C(\mathbf{x}-\mathbf{x}',t)\delta_{ij} = \langle \psi_i(\mathbf{x},t)\psi_j(\mathbf{x}',t) \rangle, \tag{6}$$

$$D(\mathbf{x}-\mathbf{x}',t)\delta_{ij} = \langle \psi_i(\mathbf{x},t)\Pi_j(\mathbf{x}',t) \rangle, \tag{7}$$

$$G(\mathbf{x}-\mathbf{x}',t)\delta_{ij} = \langle \Pi_i(\mathbf{x},t)\Pi_j(\mathbf{x}',t) \rangle, \tag{8}$$

where I have taken advantage of the symmetry among the N components which is preserved for any finite time.

For arbitrary N the equations of motion for C, D, and G do not form a closed set because of the nonlinear terms in $\partial V/\partial\psi_i(\mathbf{x})$. In the large-$N$ limit it is well known[13] that the graphical structure of the associated perturbation theory in powers of the quartic coupling simplifies considerably and one obtains the closed set of equations for the spatial Fouriertransformed quantities as follows:

$$\dot{C}=2D, \quad (9)$$

$$\dot{D}=-3HD-\Gamma C+G, \quad (10)$$

$$\dot{G}=-6HG-2\Gamma D, \quad (11)$$

where

$$\Gamma=r+uS(t)+q^2/a^2(t) \quad (12)$$

and

$$S(t)=\int[d^3q/(2\pi)^3]C(\mathbf{q},t). \quad (13)$$

These equations, together with (1), form a nontrivial coupled set of nonlinear equations.

The associated equilibrium problem at temperature T with a probability distribution $\exp-H/T$, where $H=\int d^3x\rho(\mathbf{x})$ and $a=1$, can be solved exactly (for the same reasons as in the nonequilibrium case). The average value of the Fourier transform of the momentum correlation function is given by the equipartition theorem to be $G(q)=T$, while from symmetry considerations $D(q)=0$. The Fourier transform of the matter field correlation function is given by

$$C(q)=T/(q^2+r+uS), \quad (14)$$

where the "short-range" order parameter $S=\langle\psi^2(\mathbf{x})\rangle$ is given in equilibrium by (14) with $C(\mathbf{q},t)$ replaced by $C(q)$. For very high temperatures one finds that[14] $S=T^2/(6\pi^2u)^{1/2}$. For $r<0$ there is a second-order phase transition at a temperature $T_c=(2\pi^2|r|/u)^{1/2}$. For lower temperatures the system spontaneously orders [there is a direction i for which $\langle\psi_i(\mathbf{x})\rangle\neq0$]. There are Nambu-Goldstone modes, the Fourier transform of $\langle\delta\psi_i(\mathbf{x})\delta\psi_i(\mathbf{x}')\rangle$ [where $\delta\psi_i(\mathbf{x})=\psi_i(\mathbf{x})-\langle\psi_i(\mathbf{x})\rangle$] is given by Tq^{-2}, and the short-range order parameter S is given by $|r|/u$ for all temperatures below T_c.

I have solved the coupled set of equations (9)–(13) numerically through a direct forward-step integration where, after each step, the wave-number integrations over q are carried out using Simpson's rule and a 100-point mesh. While I report the detailed results here for the case $r=-5$, $u=0.1$, $a(0)=1$, and $T_0=100$ ($T_c=10\pi$), the qualitative behavior I have found persists for other choices of parameters and initial conditions.

The time evolution of the system shows three distinct regimes. In the earliest time regime $t<0.01$ the average kinetic energy, $G(t)$, drops from its initial value of 1.68×10^6 to a value of 2.41 for $t=0.01$. The short-range order parameter $S(t)$ changes very little over this time regime from an initial value of 457.85 to 456.98 at $t=0.01$. The scale factor a has grown from 1 to $e^{3.79}$ during this period. The energy density reflects the drop in the kinetic energy as well as in the spatial energy by dropping from its initial value of 1.68×10^6 to 4.57×10^3 at time $t=0.01$. It is during the next time regime $0.01\leqslant t\leqslant15$ when the system undergoes the major portion of its inflationary motion. In Fig. 1 we see that the scale factor increases by 660 e-foldings. During this time $G(t)$ and $S(t)$ are decaying exponentially in time with apparently the same rate. I obtain the following excellent fits to the data:

$$S=S(t=0)e^{-0.145t},$$

$$G=2.469^{-0.148t}.$$

Note that S, G, and ρ are changing with a rate (0.15) *much less* than $H\sim200$. For times greater than 15, S approaches (see Fig. 1) its equilibrium value $S=|r|/u=50$, and G and ρ are slowly decaying to zero. A careful look at the data shows that one has damped oscillations for $S(t)$ in this regime corresponding to a frequency $\omega_0=(2|r|)^{1/2}=\sqrt{10}$. For $t>15$, $G(t)$ oscillates with a frequency $\omega_0/2$. For times $t>15$ the scales factor a grows much more slowly [the best fit is to a power law $a(t)=\exp(656.6)t^{1.78}$].

Because of the red-shifting terms, q^2/a^2, one finds that as $a(t)$ increases, the q dependence of $C(q)$ is quickly quenched in (becomes time independent) except for an overall multiplicative constant proportional to $S(t)$.

The basic picture we arrive at is that there is strong inflation over an intermediate time regime followed by the system locally ordering near a value of $|\psi|=(|r|/u)^{1/2}=[\langle\psi^2(x)\rangle]^{1/2}$ near the bottom of the potential well with some oscillations about the bottom of the well. The spatial structure in the problem should be viewed in terms of the physical distance scale $\mathbf{x}_p=a(t)\mathbf{x}$ and physical wave number $\mathbf{k}=\mathbf{q}/a(t)$. The associated correlation function $C_p(\mathbf{k})=a^3(t)$

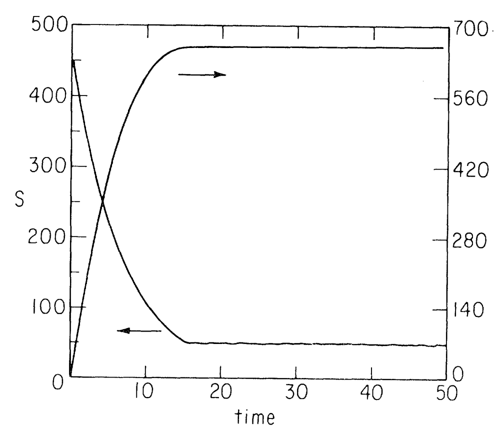

FIG. 1. The logarithm of the Robertson-Walker scale factor $a(t)$ and the short-range order parameter $S(t)$ are plotted vs time for the case $r=-5$, $u=0.1$, $a(0)=1$, and $T_0=100$.

$\times C(\mathbf{k}a(t),t)$ clearly goes over to a Bragg peak $C_p(\mathbf{k}) = (2\pi)^3\delta(\mathbf{k})m_E^2$ in the limit $a(t)\to\infty$. m_E^2 is just the "equilibrium magnetization" squared $[m_E = (|r|/u)^{1/2}]$ which is identified from

$$\int \frac{d^3k}{(2\pi)^3} C_p(k) = \int \frac{d^3}{(2\pi)^3} C(\mathbf{q},t) = S(t) \to m_E^2.$$

We also conclude that a typical domain size, which can be defined as the inverse of the width of the Bragg peak (see the discussion in Ref. 7), is just proportional to the scale factor $a(t)$. Thus for spatial scales $x \ll a(t)x_0$, the system looks as if it has long-range order.

While I have presented results for a particular set of parameters and initial conditions, there is nothing at all special about the choices I have made—nothing needs to be adjusted to obtain inflation followed by a quasistationary state. Note that the picture presented here differs considerably from the scenario presented elsewhere. I do not find a system that is initially "frozen" with $\psi(\mathbf{x},t)=0$. Instead, there are large fluctuations in $\psi(\mathbf{x},t)$. The inflationary period develops because the scale of these fluctuations does not change rapidly with time. We also note that because of the strong-nonequilibrium nature of the problem, we cannot sensibly treat the problem as being in local equilibrium at any time. Certainly the notion of an effective temperature and a sharp transition near T_c is nowhere to be found.

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[1]See, for example, the articles in *The Very Early Universe*, edited by G. W. Gibbons, S. W. Hawking, and S. T. C. Siklos (Cambridge Univ. Press, Cambridge, England, 1983).

[2]A. H. Guth, Phys. Rev. D **23**, 347 (1981).

[3]A. D. Linde, Phys. Lett. **108B**, 389 (1982), and Phys. Lett. **114B**, 431 (1982).

[4]A. Albrecht and P. J. Steinhardt, Phys. Rev. Lett. **48**, 1220 (1982).

[5]G. F. Mazenko, W. Unruh, and R. Wald, Phys. Rev. D **31**, 273 (1985). A first effort to address some of these questions is given in A. Albrecht and R. Brandenberger, to be published.

[6]A. D. Linde, Rep. Prog. Phys. **42**, 25 (1979).

[7]This model has recently been treated in a nonequilibrium context relevant to condensed-matter physics by G. F. Mazenko and M. Zannetti, Phys. Rev. Lett. **53**, 2106 (1984).

[8]I assume for simplicity that the curvature term is negligible and incorporate the factor of G, the gravitational constant, into my definition of time scale. The assumption of an isotropic and homogeneous medium is well suited to this model since, due to the continuous symmetry, it has smooth Bloch walls rather than sharp domain walls separating ordered regions.

[9]These follow from the equation $\Box\psi_i = -\partial V/\partial\psi_i$ where the invariant d'Alembertian is evaluated in the Robertson-Walker metric. See, for example, S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972), p. 163. It may be interesting to generalize this equation to include the coupling to other matter through the introduction of thermal noise and the associated damping. This is a straightforward extension which will be discussed elsewhere.

[10]A nonthermal set of initial conditions has been proposed by A. D. Linde, Phys. Lett. **129B**, 177 (1983).

[11]Thus, we never have global spontaneous symmetry breaking for any finite time. On the other hand, domain sizes corresponding to a particular ordered state are growing rapidly and it appears locally that the symmetry is broken. For an example in a condensed-matter context, see G. F. Mazenko and O. T. Valls, Phys. Rev. B **27**, 6811 (1983).

[12]It is only because we are interested in the leading term in a $1/N$ expansion which allows us to avoid the need to deal with fields at different times in (6)–(8).

[13]See, for example, S. Ma, *Modern Theory of Critical Phenomena* (Benjamin, Reading, Mass., 1976), p. 303.

[14]Compare with Eq. (3.5) in Ref. 6.