Physical Spectrum of Compactified Strings

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The single-particle states of compactified bosonic string theory are classified by a direct product of a fundamental representation of the Heisenberg algebra for the noncompact flat dimensions and an affine-g Lie algebra, where the rank of its Lie subalgebra g is the number of internal dimensions and its identity depends on their length scales. The operators that create these states are explicitly constructed in light-cone gauge. The massless particles lie in the adjoint representation of g.

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Some of the most attractive proposals for unifying the low-energy elementary-particle interactions into a single theory are formulated in $D>4$ space-time dimensions. The d "internal" dimensions ($d=D-4$ in realistic models) are difficult to observe because, for example, they form a small compact or finite-volume space at each four-dimensional space-time point. A D-dimensional field on this space is interpreted as an infinite set of $(D-d)$-dimensional fields, where each of these corresponds to a term in the harmonic expansion on the space of extra dimensions. For the case $D=5$ and $d=1$, the extra dimension is a circle, and there is one four-dimensional field for each integer N. The four-dimensional mass squared then goes as $(N/R)^2$, where R is the radius of the circle.[1]

String theories have an infinite number of states lying on Regge trajectories in $D=2$, 10, or 26. The slope of the Regge trajectory, α', provides one mass scale. Further mass scales R_I ($I=1,\ldots,d$) appear to be required for each compactified dimension, although there may eventually be some physical principle that determines the ratios R_I^2/α'. Each D-dimensional momentum eigenstate corresponds to an infinite number of $(D-d)$-dimensional states, just as for field theory. The mass spectrum of the compactified open string then has the form[2]

$$\alpha' M_{D-d}^2=\bar{n}-1+\alpha'\sum_{I=1}^{\alpha} p^I p^I, \qquad (1)$$

where $\bar{n}$ is a nonnegative integer characterizing the nonzero string modes (discussed below), and the internal-momentum components p^I are quantized in units of $1/R_I$, in analogy with compactification on a torus. (Sums on repeated contravariant indices always imply the metric tensor δ_{ij}.) For arbitrary values of R_I, these states belong to a representation of Heisenberg-U$(1)^d$, which is a direct product of Fock spaces. In the limit that $R_I^2=\alpha'$, the last term of Eq. (1) becomes an integer and the degeneracy of states at each mass level $\geqslant 0$ increases. In this paper we construct the states corresponding to a different type of compactification that requires $R_I^2=\alpha'$ and $(2\alpha')^{1/2}p^I$ to take on values in the root lattice (including the infinite number of points obtained by translating the roots) of a finite-parameter, rank-d, non-Abelian Lie algebra g, restricted to those cases where the roots have equal length. The string states become organized into representations of affine-g, where the non-Abelian Lie subalgebra g of affine-g has rank d. The Heisenberg-U$(1)^d$ is a subalgebra of affine-g. The first excited level of massless particles carry the adjoint representation of g. The internal symmetry is then generated in a new way. It is identified as an unexpected symmetry of compactified strings, neither as isometries of an internal space nor as explicit internal symmetry in higher dimensions.

The algebraic structure underlying the spectrum is affine-g, a Kac-Moody Lie algebra.[3] The construction and the algebraic structure of the light-cone-gauge, physical-particle operators for the compactified bosonic string are found in Eqs. (11) and (12) below. We apply these general results to the $E_8\otimes E_8$ "heterotic" string[4] by calculating the first few levels of this $D=26$, $d=16$ model (see Table I). Supersymmetric generalizations are being studied presently.[5]

We approach the problem in two ways: First, we show by a simple counting argument, which will prove useful in practical calculations of spectra, how the on-mass-shell physical states at each integer value of $\alpha' M_{D-d}^2$ are organized into representations of g; then, using a physical light-cone-gauge version of the Frenkel-Kac construction,[6] we rewrite the physical operators in a form that explicitly displays the affine-g structure of the theory.

The physical (on-mass-shell) states of the $D=26$ bosonic string are created by direct products of the light-cone string operators A_n^i ($i=1,\ldots,D-2$) and $\exp(ip^i\hat{x}^i)$ on the ground state. In light-cone gauge

TABLE I. The first four levels of the $E_8 \otimes E_8$ compactified open bosonic string. In the closed-string $E_8 \otimes E_8$ version of Ref. 4, the tachyon is decoupled and $\alpha' M_{I0}^2$ is 4 times that given in the table.

$\alpha' M_{I0}^2$	No. states	$E_8 \otimes E_8$ representation labeled by dimensionality
−1	1	(1,1)
0	496	(248,1) + (1,248)
1	69 752	(3875,1) + (1,3875) + (248,248) + (248,1) + (1,248) + 2(1,1)
2	2 115 008	(30 380,1) + (1,30 380) + (3875,248) + (248,3875) + 2(248,248) + (3875,1) + (1,3875) + 3(248,1) + 3(1,248) + 2(1,1)
3	34 670 620	(147 250,1) + (1,147 250) + (27 000,1) + (1,27 000) + (30 380,248) + (248,30 380) + (3875,3875) + (30 380,1) + (1,30 380) + 2(3875,248) + 2(248,3875) + 3(3875,1) + 3(1,3875) + 5(248,248) + 5(248,1) + 5(1,248) + 5(1,1)

$A_n^+ = 0$ and A_n^- are dependent operators, and the zero-mode operator $\hat{p}^i$ is identified by $(2\alpha')^{1/2} \times \hat{p}^i = A_0^i$.[7] (The A_n^i are also called DDF operators.[8]) The operators $\hat{x}^i$ satisfy canonical commutation relations with $\hat{p}^j$. The $(D-d)$-dimensional mass operator is

$$M_{D-d}^2 = 2\hat{p}^+\hat{p}^- - \sum_{i=1}^{D-d-2} \hat{p}^i\hat{p}^i, \tag{2}$$

where the sum on i is over the noncompactified, transverse directions. It follows from the light-cone-gauge commutation relations that

$$[\alpha' M_{D-d}^2, A_n^i] = -nA_n^i. \tag{3}$$

Note that A_n^i commutes with $\hat{p}^j$, so that only $\exp(ip^i\hat{x}^i)$ creates momentum in the transverse directions. It is easily confirmed for the zero-mode subspace that

$$\alpha' M_{D-d}^2|p\rangle = \left(\alpha' \sum_{I=1}^{\alpha} p^I p^I - 1\right)|p\rangle, \tag{4}$$

where the sum is on internal dimensions only and $|p\rangle = \exp(ip^I\hat{x}^I)|0\rangle$. Since the set of operators with internal indices commutes with the set of operators with noncompact transverse indices, we can separate the problem into two pieces. The Heisenberg algebra of dual theory is defined by

$$[A_n^i, A_m^j] = n\delta^{i,j}\delta_{n,-m} \tag{5}$$

plus the commutation relations of the transverse zero-mode operators. The operators with noncompact-space labels alone create the Fock-space states of the $(D-d-2)$-dimensional Heisenberg subalgebra, where the states carry a $(D-d)$-momentum label and the occupation-basis labels. We can then apply the internal-space operators to each of these states to obtain the complete set of physical states.

We study the degeneracy patterns of states of fixed $\alpha' M_{D-d}^2$. The product of A_{-n}^I operators contributes to $\bar{n}$ of Eq. (1), while the $\exp(ip^I\hat{x}^I)$ operators contribute to the last term. Since the A_n^I form a d-dimensional Heisenberg algebra, the number of independent operator combinations is given by the generating function (which is the inverse of the dth power of the Euler function[9]),

$$\prod_{k=1}^{\infty}(1-x^k)^{-d} = \sum_{n=0}^{\infty} M_n(d)x^n = 1 + dx + (d/2)(d+3)x^2 + (d/6)(d+1)(d+8)x^3 + (d/24)(d+1)(d+3)(d+14)x^4 + \ldots, \tag{6}$$

where $M_n(d)$ is the degeneracy of the nth state of the d-dimensional Heisenberg algebra representation. If we now require the $\alpha' M_{D-d}^2 + 1$ to be nonnegative integers, we must require the last term in Eq. (1) to be a nonnegative integer; given a scheme to do this, the total degeneracy results from convoluting these numbers with the degeneracy given by Eq. (6).

The connection with the algebra g comes from solving this problem by imposing

$$R_I^2 = \alpha', \tag{7}$$

and requiring $(2\alpha')^{1/2}p^I$ to be any point on the infinite root lattice of a semisimple non-Abelian finite algebra with roots of equal length. [If the roots of the algebra do not have the same lengths, then the particles in the irreducible representations of g do not have equal $(D-d)$-dimensional masses.] The Lie algebras with roots of equal length are A_n [the algebra of SU$(n+1)$], D_n $(n>2)$ [the algebra of SO$(2n)$], E_6, E_7, and E_8, and in the tree approximation, g can be any direct product of these simple factors with rank d.

The root space of a semisimple Lie algebra is a Euclidean space. Even so, in order to specify the momentum components by integers when all $R_I^2=\alpha'$, it is convenient to use the simple roots $\boldsymbol{\alpha}_L$ (specified by the Dynkin diagram) as a nonorthogonal basis of the root lattice with metric given by the Cartan matrix with elements $C_{LM}=(\boldsymbol{\alpha}_L,\boldsymbol{\alpha}_M)$, which is defined in terms of the orthonormal coordinates where the system was quantized as $\sum_{I=1}^{d}\alpha_L^I\alpha_M^I$.[10] The internal-momentum components p^I are then

$$(2\alpha')^{1/2}p^I=\sum_{L=1}^{\alpha}N^L\alpha_L^I, \tag{8}$$

where $\boldsymbol{\alpha}_L$ is a simple root normalized for each L to $(\boldsymbol{\alpha}_L,\boldsymbol{\alpha}_L)=2$ and the N^L are integers. We rewrite the zero-mode operators in the dual basis as $\hat{x}_L=(\hat{\mathbf{x}},\boldsymbol{\alpha}_L)/(2\alpha')^{1/2}$, and $\hat{p}_M=(2\alpha')^{1/2}(\mathbf{p},\boldsymbol{\alpha}_M)$, so that

$$[\hat{x}_L,\hat{p}_M]=iC_{LM}. \tag{9}$$

If we write the momentum components in the basis dual to the simple roots, defined by $a_L=(2\alpha')^{1/2}(\mathbf{p},\boldsymbol{\alpha}_L)=N^MC_{ML}$, then Eq. (1) is

$$\begin{aligned}\alpha'M_{D-d}^2&=\bar{n}-1+\tfrac{1}{2}N^LC_{LM}N^M\\&=\bar{n}-1+\tfrac{1}{2}a_LC^{LM}a_M\\&=\bar{n}-1+\tfrac{1}{2}(a,a),\end{aligned} \tag{10}$$

where the Dynkin labels a_L are always integers, and for groups with roots of equal lengths, (a,a) is an even integer for all weights $\mathbf{p}$, which are of the form of Eq. (8). [Note that now the zero-mode state can be written $|\{a\}\rangle=\exp(ia_L\hat{x}^L)|0\rangle$.]

We take the simple example of $d=1$, so that $\alpha'M^2=\bar{n}+N^2-1$. The solutions for zero mass are $\bar{n}=1$, $N=0$, and $\bar{n}=0$, $N=\pm1$, and so there are three states with internal momenta $\sqrt{2}$, 0, and $-\sqrt{2}$, an SU(2) triplet. For $\alpha'M^2=1$, there are four solutions with momenta $\sqrt{2}$, 0, 0, $-\sqrt{2}$, an SU(2) singlet plus triplet, and so on. In this way we build up the representation of affine SU(2) with the singlet as highest weight[11]; no spinors appear in this representation.

For larger values of d, the set of weights $\{w\}$ in a Weyl orbit (which is a set of weights related by Weyl reflections, which leave the root lattice invariant) all have the same integer values of $(w,w)/2$. If we recall that each and every Weyl orbit has one and only one weight for which the Dynkin labels are nonnegative integers, then from Eq. (5) we can compute the degeneracy of each weight and very quickly regroup the orbits into representations of g.[12] It is not difficult to apply this method to quite complicated cases, such as $D=26$, $d=16$, with $g=E_8\otimes E_8$. The results for the open string up to the fourth level are listed in Table I. The physical states of the heterotic string are given by direct products of right-moving, $N=1$ superstring states and the left-moving bosonic states consisting of the standard SO$(D-d-2)$ multiplets of the Heisenberg-U$(1)^{D-d-2}$ algebra, each of which is the highest weight of the $c=\frac{1}{2}$ representation of affine $E_8\otimes E_8$ given in the table. The squared masses of the heterotic closed-string states are four times those given in the table, and the tachyon is decoupled due to a constraint condition.

We have made it plausible that for special choices of the lattice, Eq. (8), of internal-momentum values, the string states are organized into representation of an algebra g of rank d. In fact all these states lie in irreducible representations of affine g. We now transform the operators A_n^I and $\exp(ip^I\hat{x}^I)$ into a set of affine operators that generates the same Hilbert space, which is now seen to be an infinite tower of g representations. The construction is similar to the one given by Frenkel and Kac using the covariant vertex operator.[6] However, we construct the affine generators only for the internal dimensions and insist on using the light-cone string operators so that the states constructed by application of these operators to the ground state will be on the $(D-d)$-dimensional mass shell and have positive norm.

The Frenkel-Kac construction uses the moments of the vertex operator, which can create all physical states. Thus we define the operators

$$A_n(r)=\sum_{I=1}^{\alpha}r^IA_n^I, \tag{11a}$$

$$X_n(r)=\frac{1}{2\pi i}c_r\oint dz\,z^n\!:\exp\left[i\frac{r^IQ^I(z)}{(2\alpha')^{1/2}}\right]:, \tag{11b}$$

where normal ordering means that A_n is to the right of A_{-n} $(n>0)$ and $\hat{p}^I$ is to the right of $\hat{x}^I$; $Q^I(z)$ here is defined in terms of the light-cone operators,

$$Q^I(z)=\hat{x}^I-2i\alpha'\hat{p}^I\ln(z)+i(2\alpha')^{1/2}\sum_{\substack{n\neq0\\n=-\infty}}^{\infty}\frac{1}{n}A_n^Iz^{-n}, \tag{11c}$$

where the roots r are all normalized to $(r,r)=2$. The factor c_r in Eq. (11b) satisfies $c_rc_s=(-1)^{rs}c_sc_r$, $c_rc_{-r}=1$, and $c_rc_s=\epsilon(r,s)c_{r+s}$ $(r+s\neq0)$, where $\epsilon(r,s)=\pm1$.[6]

With these definitions, it is a standard computation to obtain the algebraic equations satisfied by the operators $X_n(r)$ and $A_n(s)$ (Ref. 6 and Frenkel[13] and Goddard and Olive[14]) as follows:

$$[A_n(s),X_m(r)]=(s,r)X_{n+m}(r), \tag{12a}$$

$$[X_n(r),X_m(s)]=\begin{cases}\epsilon(r,s)X_{n+m}(r+s), & \text{if } r+s \text{ is a root and } r\neq -s,\\ 0, & \text{if } r+s \text{ is not a root, but not zero,}\end{cases} \tag{12b}$$

$$[X_n(r),X_m(-r)]=A_{n+m}(r)+2cn\delta_{n,-m}, \tag{12c}$$

$$[A_n(r),A_m(s)]=2cn(r,s)\delta_{n,-m}. \tag{12d}$$

[The factors c_r in Eq. (11b) are needed for Eq. (12b) to be a commutator when $rs=\pm 1$.] This is the affine Kac-Moody Lie algebra of g (Refs. 3 and 15), where the roots of g have equal length. Of special importance are the terms with $c\neq 0$ in Eqs. (12c) and (12d), called the central extension. (In the bosonic string construction, $c=\frac{1}{2}$.) The physical states of the compactified string are in representations of affine-g.

It is easily confirmed that the $X_n(r)$ create and destroy on-mass-shell states. By direct computation,

$$[\hat{p}^-,X_n(r)]=-(n/2\alpha' p^+)X_n(r), \tag{13a}$$

$$[\alpha' M^2_{D-d},X_n(r)]=-nX_n(r), \tag{13b}$$

$$[(2\alpha')^{1/2}\hat{p}^I,X_n(r)]=r^IX_n(r), \tag{13c}$$

where we have used the light-cone gauge commutators,

$$[\hat{p}^-,A^i_n]=(n/2\alpha' p^+)A^i_n, \tag{14a}$$

$$[\hat{p}^-,\hat{x}^I]=-(i/p^+)\hat{p}^I. \tag{14b}$$

(Recall that $\hat{p}^-$ is a dependent operator.)

Equation (12) is valid when r and s are roots. However, using the counting arguments above, we can still classify the states at each mass level for the $\mathrm{Spin}(32)/Z_2$ weight lattice of the SO(32) heterotic string in terms of the representations of $\mathrm{Spin}(32)/Z_2$ (the representations congruent to the adjoint and one spinor).

In summary, if $R_I^2=\alpha'$ and the momentum vectors lie on the root lattice of a semisimple Lie algebra g of rank d, each state made from the noncompactified space operators is the highest weight of the basic representation of affine-g. These states are linear combinations of the physical states constructed from the light-cone operators A^I_n and $\exp(ip^I\hat{x}^I)$, and we have confirmed that they do couple in general amplitudes. This kind of analysis should be helpful in achieving more realistic string models. Of course, the internal consistency of such compactified string theories regarding loop amplitudes, unitarity, absence of tachyons and anomalies, and finiteness must be checked.

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