Comment on "Eden Model in Many Dimensions"

In a recent Letter Parisi and Zhang[1] studied the Eden model in d dimensions for large d and, using the results of a $1/d$ expansion, argued that there might exist an upper critical dimension above which the clusters are not compact. However, it has been proved by Richardson[2] that for Eden clusters with N sites in d dimensions, the diameter varies as $N^{1/d}$, and the surface as $N^{1-1/d}$ for large N at a fixed (integer) d. There is no upper critical dimension. A simplified version of Richardson's argument is given below. The reason for the discrepancy with Parisi and Zhang is discussed.

The discrete-time Eden process is related to a continuous-time Markovian epidemic process defined[2] as follows: Each site of a d-dimensional hypercubical lattice can exist in any one of two possible states, infected or healthy. At time $t=0$, all sites except the origin are healthy. In a small time interval dt, the probability that a healthy site becomes infected is dt if it has at least one infected neighbor, 0 otherwise. An infected site never recovers. Clearly, the conditional probabilities of getting different allowed $(n+1)$-site clusters from a given n-site infected cluster in this process are the same as in the Eden process.

Let t_N be the time taken by the infection to reach N sites, and S_N be the (random) number of perimeter sites (healthy sites neighboring infected sites) of the N-site cluster. Clearly

$$\langle t_{N+1}\rangle - \langle t_N\rangle = \langle 1/S_N\rangle, \tag{1}$$

where the angular brackets denote expectation values. On a d-dimensional hypercubical lattice S_N is never less than $2d\,N^{1-1/d}$, and hence from Eq. (1)

$$\langle t_{N+1}\rangle \leq \tfrac{1}{2}(N^{1/d}-1)+(2d)^{-1}, \quad \text{for } N\geq 1. \tag{2}$$

With use of the mutual independence of $t_{i+1}-t_i$ for different i, it is easy to show that $(\Delta t_N)^2$, the variance of t_N, is given by

$$(\Delta t_N)^2 = \sum_{i=1}^{N-1}\langle s_i^{-2}\rangle \leq \sum_{i=1}^{N-1} i^{-2+2/d}(2d)^{-2}.$$

It follows that Δt_N is finite except if $d=2$ when it can increase at most as $(\log N)^{1/2}$. Thus, fluctuations in t_N are small, and $\Delta t_N/\langle t_N\rangle$ tends to zero for large N.

For the epidemic process it is easy to prove[3] that the average velocity V with which the boundary between the healthy and infected sites moves outwards is bounded from above. For the d-dimensional hypercubical lattice, Hammersley's argument[3] gives

$$V \leq (2d-1)e = V_{\max}. \tag{3}$$

The probability that a site at a distance greater than $V_{\max}t$ from the origin is infected at time is exponentially small for large times t. (We measure distance along bonds of the lattice.) This implies that for large N with probability 1,

$$t_N \geq (d!N)^{1/d}2\,V_{\max}. \tag{4}$$

Since both the upper and lower bands on t_N vary as $N^{1/d}$, we see that $\langle t_N\rangle$ varies as $N^{1/d}$. The mean cluster size is $V\langle t_N\rangle$, which also varies as $N^{1/d}$ for large N. Hence the clusters are compact for any d.

The average number of perimeter sites per unit cross-sectional area of the surface is a measure of the roughness of the surface and is equal to the propagation speed of the wave front, and hence by Eq. (3) stays less than $(2d-1)e$ for large N. Hence $\langle S_N\rangle$ must vary as $N^{1-1/d}$ for large N.

The analysis of Parisi and Zhang is valid so long as the typical coordination number of a site ($\sim \log N$ for the origin) is much less than the maximum coordination number $(2d)$ of the lattice. In this limit all sites are surface sites, and the behavior of Eden clusters is quite different from the large-size limit ($N^{1/d} \gg 1$, i.e., $\log N \gg d$) when the fractional number of surface sites in a cluster is negligible.

The different behavior Eden clusters in the small-N regime ($\log N \lesssim d$) from the large-N limit ($\log N \gg d$) suggests that one must be very careful in extrapolations of small-N data to the infinite-N limit in other cluster studies also. The maximum value of $\log N$ reachable by exact enumerations is only about 4, and by Monte Carlo simulations about 15. Hence, these would be expected to show strong finite-size effects.

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