Resistance of Small Metallic Loops

R. Landauer and M. Büttiker
IBM Thomas J. Watson Research Center, Yorktown Heights, New York 10598
(Received 5 February 1985)

An earlier analysis of small one-dimensional rings of normal metal, driven by an external time-dependent magnetic flux, is extended to allow for inelastic scattering. This supplements the Josephson-type behavior of the earlier discussion by dissipative effects. As in the case of strong one-dimensional localization, dc current grows as inelastic scattering is first introduced, but diminishes again with sufficiently intense inelastic scattering.

PACS numbers: 72.15.Lh, 73.60.Dt

A one-dimensional loop of normal metal, at zero temperature, with elastic scattering and *without* leads, driven by a time-dependent magnetic flux, was analyzed by Büttiker, Imry, and Landauer.[1] The work was extended in a series of papers concerned with loops *with* leads.[2] Here, we return to the unconnected loop, supplementing the analysis of Ref. 1 by including inelastic scattering. Reference 1 pointed out that the single-electron states of such a ring can be obtained from the band structure of a crystal with $\mathscr{V}(x) = \mathscr{V}(x+L)$, where $\mathscr{V}(x)$ is the potential around the loop with circumference L. The electronic states of the loop are obtained from the bands, $U_n(k)$, of the periodic potential via the rule $k = -(2\pi/L)\Phi/\Phi_0$, where $\Phi_0 = hc/e$ is the single-electron flux quantum. For each flux Φ, there is a ladder of states, $U_n(\Phi)$, shown in Fig. 1. The electronic states are periodic in flux, with period Φ_0. For a time-independent flux Φ, we find a zero-temperature current $j = -(e/L)\sum_n v_n = c\sum_n \partial U_n/\partial\Phi$. The summation includes all occupied states, up to the Fermi energy E_F. Thus, Ref. 1 predicts a persistent current, which is a periodic function of the flux, with period Φ_0. In the presence of a flux which increases linearly in time, the induced electromotive force $E = (1/cL)\,d\Phi/dt$ drives the ladder of states through the Brillouin zone, according to $\hbar\dot{k} = -eE$. If we assume that E is small enough so that Zener tunneling between "bands" is negligible, then the field produces an oscillating current with frequency $\omega = eV/\hbar$, i.e., a Josephson frequency with a single electronic charge. Here we have taken $V = EL$. The time-average current vanishes. The same closed loop *without leads* was analyzed in a set of later papers,[3,4] subsequent to Ref. 1. References 3 and 4 differ from Ref. 1 in several aspects, but particularly in the fact that they ascribe a resistance to a loop, described as a closed Hamiltonian system. References 3 and 4 also stress a flux period $hc/2e$, in contrast to the period hc/e of Refs. 1 and 2. The period hc/e in a single loop, presumably small compared to the inelastic mean free path, has been observed.[5]

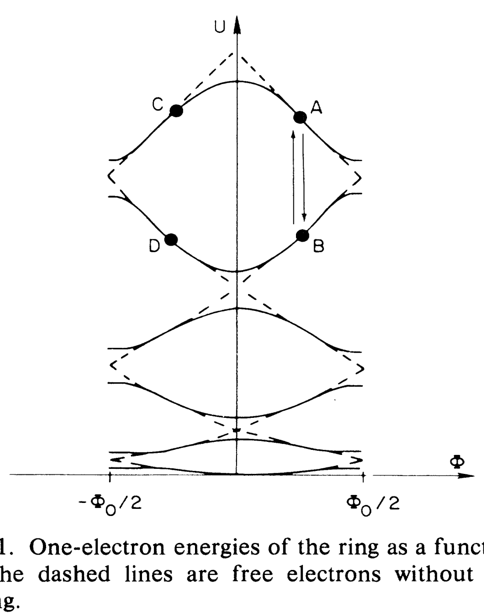

FIG. 1. One-electron energies of the ring as a function of flux. The dashed lines are free electrons without elastic scattering.

The analysis in Ref. 1 (and Refs. 3 and 4) was limited to zero temperature and to a *strictly* one-dimensional loop. The electrons are only scattered elastically and uncoupled from other degrees of freedom which produce energy loss and incoherence. Our system can store energy, but cannot dispose of it. As a result, we do not have ordinary resistive behavior. In this note, we consider the one-dimensional loop with inelastic scattering and show how resistive behavior arises. We are clearly making a conceptual point, not applicable in quantitative detail to experimentally realizable systems. We therefore concentrate on the most easily analyzable cases, and avoid complex formalism.

We assume that the coupling of the electrons to other degrees of freedom is sufficiently weak that inelastic scattering can be viewed as a source of transitions between the states shown in Fig. 1. For simplicity we concentrate on the case where kT is smaller than the spacing of the two bands adjacent to the Fermi level, and also assume that an even number of electrons are present to fill the spin-degenerate levels. Thus, we have an Nth band which is relatively fully occupied and a $(N+1)$th band which is almost empty, and need not be concerned with other levels which are either totally full, or empty. Successive higher-lying bands will have alternating signs in their current contributions. In general, however, the current will increase as we go up in energy, and at zero temperature the overall sign will be determined by the highest-lying band. (If we depart from the case we are emphasizing, and consider many levels in a range kT at the Fermi level, then there will, obviously, be much more effective cancellation). We also concentrate on the case where the band which is almost full has its minimum energy at zero flux, whereas the next higher band has a maximum there. Consider, first, a time-independent flux. Inelastic scattering causes transitions from the Nth state to the $(N+1)$th state. If the self-inductance of the loop is very small, then the flux through the loop remains constant during these transitions. Therefore, the electron makes transitions between the states $U_N(\Phi)$ and $U_{N+1}(\Phi)$ belonging to the same flux, or the same value of $k=-(2\pi/L)\Phi/\Phi_0$. If the electron is in the lower state, then the current in the ring, for each spin direction, is $j_1=-(e/L)\sum_0^N v_n$; if the electron is in the level above the Fermi energy, then the current is $j_2=-(e/L)(\sum_0^{N-1}v_n+v_{N+1})$. Inelastic transitions, therefore, cause fluctuations in the persistent current, but do not destroy it. The average persistent current is determined by the equilibrium occupation probabilities of the two levels involved, $\rho_{1,\mathrm{eq}}$ and $\rho_{2,\mathrm{eq}}$, and is given by $j=-(e/L)[\sum_0^n v_n+(v_{N+1}-v_N)\rho_{2,\mathrm{eq}}]$, where we have taken into account that $\rho_{1,\mathrm{eq}}+\rho_{2,\mathrm{eq}}=1$. [Note that in this case, where the occupation of the Nth and $(N+1)$th states are related, the occupation probabilities of these one-electron states are also probabilities for the occurrence of the corresponding total state, in which all lower $N-1$ states are fully occupied.] Compared to the current at $T=0$, a small rise in temperature causes only an exponentially small decrease in the persistent current.

Consider a time-dependent flux causing the electron states to cycle to the right, through their respective "bands," in Fig. 1. Reference 1 ignored relaxation, via inelastic transitions, between the levels of Fig. 1, e.g., between A and B. In the absence of such transitions each "band" in Fig. 1 has a time-dependent probability of occupation. We assume that the relaxation in the occupation probability between the two levels of concern can be characterized via

$$\partial\rho/\partial t=-\tau^{-1}[\rho(t)-\rho_{\mathrm{eq}}(\Phi(t))], \qquad (1)$$

with τ a phenomenological relaxation rate, and ρ_{eq} the equilibrium distribution toward which the relaxation proceeds. ρ_{eq} is determined by the time-dependent set of energy levels of Fig. 1. In the limit of large τ, and time-independent occupation probabilities, the time-average current contributed by each "band" vanishes. There is, then, no dc current in the presence of a linearly increasing flux $\Phi(t)$, equivalent to a dc field. Ordinary conductive dissipation is absent. In the other extreme of very short τ, $\rho=\rho_{\mathrm{eq}}(t)$. The time-average current again vanishes; contributions from A and B in Fig. 1 will cancel those from the opposite side at C and D. The oscillatory current, under constant $d\Phi/dt$, will still be present. Indeed, if the energy levels are far apart compared to kT, the oscillatory current will not be very different from that for large τ.

Our main purpose is to examine the dc current, and associated dissipation, for intermediate values of τ. In that case the distribution between A and B in Fig. 1 will exhibit some influence of the past history, and of the larger energy gap present at $\Phi=0$. Thus, A will be underpopulated and B overpopulated, relative to the actual energy spacing between A and B. We therefore have a positive contribution to the time-average particle velocity, relative to that obtained in the case of $\rho=\rho_{\mathrm{eq}}(t)$. Similarly at C and D the population will show the effect of the smaller gap at $\Phi=-\Phi_0/2$ and C will be overpopulated, relative to D. The velocity at C is positive. Therefore, we once again have a positive contribution to the time-average velocity distribution. This is our basic effect: Introducing a limited relaxation rate establishes a dc current, and a dissipation.

A dc current which vanishes at $\tau=0$ and $\tau=\infty$, and peaks in between, should not be totally unexpected. There are other, related, problems in which low-frequency transport peaks at an intermediate relaxation rate. Thermally activated escape from a metastable potential well is one example.[6] Another example comes from one-dimensional localization.[7] In the absence of inelastic effects the carriers are confined to localized states. In the presence of intense inelastic scattering, current flow is impeded by the inelastic scattering. A maximum current flow can be expected for an intermediate scattering rate.

We now make the preceding more quantitative. The dc current under constant $d\Phi/dt$ is given by

$$j=-(2e/\hbar L)\langle\rho_1 dU_1/dk+\rho_2 dU_2/dk\rangle, \qquad (2)$$

where the angular brackets denote time averaging, and the factor 2 arises from the spin degeneracy. ρ_1 and ρ_2 are the occupation probabilities of the two bands involved, with 2 the higher-lying band, and $\rho_1+\rho_2=1$. Using this in Eq. (2), together with $\langle dU_1/dk\rangle=0$, we can put Eq. (2) in the form

$$j=-(2e/\hbar L)\langle(d\Delta U/dk)\rho_2\rangle, \qquad (3)$$

where $\Delta U=U_2-U_1$. Let ω be the angular frequency with which the bands in Fig. 1 are traversed,[1] of magnitude $|eEL/\hbar|$, where E is the accelerating field in the loop. Then ΔU can be written as a Fourier series,

$$\Delta U=\sum_0^{\infty}\Delta_n\cos n\omega t, \qquad (4)$$

where we have assumed that at $t=0$ we will be at a zone center. One-dimensional bands are always of the form shown in Fig. 1, increasing or decreasing monotonically from an extremum at $k=0$. This is the central feature that simplifes the one-dimensional case and permits this analysis. A complicated shape for the potential $\mathscr{V}(x)$ will be reflected, as exemplified by perturbation theory, through a lack of correlation between successive bands and gaps, rather than by a complex shape for a particular band. Thus, the bands can be expected to have a smooth form; Δ_n is likely to diminish rapidly with n. Equation (4) yields

$$d\Delta U/dk = -\left(\sum_0^\infty n\omega\Delta_n \sin n\omega t\right)(dt/dk)$$
$$= L\sum_0^\infty n\Delta_n \sin n\omega t, \quad (5)$$

where $dk/dt = -eE/\hbar$ is given by Bloch's theorem, and is a constant during our period of concern. ρ_2 will be a solution of Eq. (1). $\rho_{2,\mathrm{eq}}(t)$ will also be given by a Fourier series,

$$\rho_{2,\mathrm{eq}}(t) = \sum_0^\infty \rho_n \cos n\omega t. \quad (6)$$

Now, however, in view of the exponential dependence of ρ on energy gaps, we can expect a $\rho_{2,\mathrm{eq}}(t)$ which peaks sharply at those instants where the two bands are closest. Thus, ρ_n cannot be expected to fall off rapidly with n. Equation (6) substituted into Eq. (1) yields the solution

$$\rho_2 = \sum_0^\infty \frac{\rho_n}{1+n^2\omega^2\tau^2}(\cos n\omega t + n\omega\tau \sin n\omega t). \quad (7)$$

In the evaluation of $\langle \rho_2 d\Delta U/dk\rangle$ in Eq. (3), the cosine terms of Eq. (7) will drop out. Using $\langle \sin^2 n\omega t\rangle = \frac{1}{2}$, we find from Eqs. (3), (5), and (7) that

$$j = -\frac{e}{\hbar}\sum_{n=1}^\infty \rho_n\Delta_n \frac{n^2\omega\tau}{1+n^2\omega^2\tau^2}. \quad (8)$$

ω and E are proportional. For small E, Eq. (8) predicts a current proportional to the field. Successive terms in the sum peak at $n\omega = 1/\tau$. In view of the dominance of low n in Eq. (4), we can expect the overall sum to have its peak at $\omega \sim 1/\tau$. At higher frequencies the dissipative effects disappear again, as indicated in the earlier, more qualitative discussion. Figure 2 gives a qualitative sketch of the dc current-voltage characteristic. In the presence of a time-independent flux we have a persistent current flowing in the loop, giving rise to the branch at $V = -(1/c)\partial\Phi/\partial t = 0$. This current reaches a maximum magnitude at points roughly halfway between $\Phi = 0$ and $|\Phi| = \Phi_0/2$, with the maximum magnitude corresponding to the limits

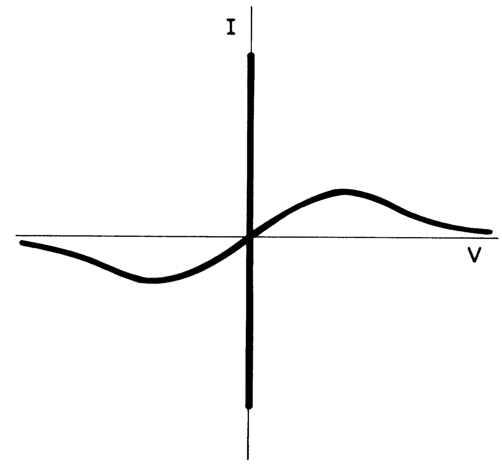

FIG. 2. Current-voltage characteristic of a small loop, showing only the dc component of the current. The branch at $V=0$ results from a thermal equilibrium distribution in Fig. 1, for fixed values of Φ. The branch with a nonzero voltage is characterized by Eq. (8) and specifies the dc current accompanying the Josephson-type oscillations.

of the branch at $V=0$. A linearly increasing flux $V = -(1/c)\partial\Phi/\partial t = \hbar\omega/e$ produces Josephson oscillations. Inelastic scattering causes these to produce the dc current in Eq. (8). As a function of the induced voltage, $V = \hbar\omega/e$, this current reaches a maximum at a voltage $V \approx \hbar/e\tau$. The slope of the resistive branch for small voltages is given by the low-frequency limit of Eq. (8). This yields a conductance

$$G = j/V = (e^2\tau/\hbar^2)\sum_{n=1}^\infty n^2\rho_n\Delta_n. \quad (9)$$

Let us define an effective mass, $m^{*-1} = \hbar^{-2}\langle d^2U/dk^2\rangle$, in analogy with ordinary band theory. The averaging, now, is intended not only as a time average, but also over the two top bands, weighting each band with its time-dependent probability of occupation, in thermal equilibrium. Equation (9) then becomes $\sigma = ne^2\tau/m^*$, with $n = 2/L$, which is the spatial density of the two electrons in our topmost bands. In fact, Eq. (1) in the case of short τ leads directly to $\rho - \rho_{\mathrm{eq}} = -\tau d\rho_{\mathrm{eq}}/dt$. This, in turn, yields $\sigma = ne^2\tau/m^*$, without the further assumptions needed for Eq. (9), regarding the relationship between kT and Fig. 1, or the need for an even number of electrons. (Note that in this relationship, varying n and $1/m^*$ by the inclusion or omission of totally filled levels has no effect on their product.) This Drude formula has, of course, only a formal analogy to the more conventional version; m^* is defined by the band structure in Fig. 1, and is not the effective mass of the original atomic lattice.

We have discussed the one-dimensional case; the case where the conductor has a real transverse cross section remains to be understood. The general features, however, involving a gradual onset of dissipation with increasing inelastic scattering, and its disappearance again with sufficiently intense scattering, can be expected to reappear. For sufficiently intense inelastic scattering even our one-dimensional approach will fail. When $\hbar/\tau$ exceeds the energy-level separation in Fig. 1, that structure will cease to be relevant. But that can be at values of τ very short compared to $1/\omega$, so that there can easily be opportunity for the maxima we have discussed to show up. This failure, at very short τ, is analogous to the irrelevance of m^* and band structure in ordinary solid-state transport theory, for extremely intense inelastic scattering.

We are indebted to Y. Imry for stimulating discussions.

[1]M. Büttiker, Y. Imry, and R. Landauer, Phys. Lett. **96A**, 365 (1983).

[2]Y. Gefen, Y. Imry, and M. Ya. Azbel, Phys. Rev. Lett. **52**, 139 (1984); M. Büttiker, Y. Imry, and M. Ya. Azbel, Phys. Rev. A **30**, 1982 (1984); M. Büttiker, Y. Imry, R. Landauer, and S. Pinhas, Phys. Rev. B (to be published), compare effects with a flux period hc/e with effects characterized, instead, by a period $hc/2e$.

[3]J. P. Carini, K. A. Muttalib, and S. R. Nagel, Phys. Rev. Lett. **53**, 102 (1984).

[4]D. A. Browne, J. P. Carini, K. A. Muttalib, and S. R. Nagel, Phys. Rev. B **30**, 6798 (1984).

[5]R. A. Webb, S. Washburn, C. P. Umbach, and R. B. Laibowitz, to be published.

[6]M. Büttiker, E. P. Harris, and R. Landauer, Phys. Rev. B **28**, 1268 (1983).

[7]D. J. Thouless in *Physics in One Dimension,* edited by J. Bernasconi and T. Schneider, (Springer, Heidelberg, 1981), p. 306.