Critical Ising Spin Dynamics on Percolation Clusters

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Ferromagnetically interacting Ising spins are placed on a fractal network (such as a critical percolation cluster) with $T_c = 0$, and endowed with a single-spin-flip dynamics. At low temperatures the collective dynamics is determined by thermal activation over energy barriers. The barrier to overturning of the spins in a domain of size L is proportional to $Z \ln L$, where Z is a new geometrical parameter characterizing the fractal. A new "singular" dynamic scaling is found in which the effective dynamic critical exponent *diverges* at criticality.

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In recent years, there has been much study of physical problems in a space which, instead of being Euclidean, is a fractal lattice, such as a percolation cluster.[1] This has included the study of the equilibrium properties of the ordering of spin systems[2]—a problem which is inherently *nonlinear*—and also of linear *dynamic* problems, such as random walks,[3,4] dielectric response,[5] elastic modes,[3,4] and spin waves.[5]

However, there had been no study of the critical dynamics near the ordering temperature of a fractal lattice, a nonlinear *and* dynamic phenomenon. Recently, Aeppli, Guggenheim, and Uemura[6] addressed this problem by an inelastic neutron scattering experiment on $Rb_2(Mg_{0.41}Co_{0.59})F_4$, where the magnetic Co ions are at percolation on a two-dimensional lattice. They fitted a relaxation time τ by the standard dynamic scaling form,[7] $\tau(T) \sim \xi_T(T)^z$, with an exceptionally large value of z (here ξ_T is the thermal correlation length). Very recently, theoretical approaches[6,8-10] have appeared which *assume* this form and then proceed to obtain estimates of z.

In this Letter, I consider the behavior of ferromagnetic Ising systems on finitely ramified fractal lattices, such as the incipient infinite cluster in percolation ("percolation cluster" for short, or PC). Other examples are Sierpiński gaskets[11] or PC's on a Bethe lattice.[12] This analysis turns out to be unexpectedly simple since this system has a zero-temperature phase transition and this means that the most divergent factors in $\tau(T)$ can be derived by consideration of thermal activation over *energy* barriers resulting from the nonuniform geometry. As a function of the length scale L, the barrier $E_{\max}$ grows *logarithmically:*

$$E_{\max}(L)/2J \cong Z \ln L + \text{const} \quad (\text{as } L \to \infty). \tag{1}$$

Here Z is a new, "universal" parameter of the fractal. I also show that (1) implies a "singular" dynamic scaling, in which $\tau(T)$ diverges as $\exp(\text{const}/T^2)$, faster than any power of $\xi_T(T)$.

I start with a Hamiltonian describing classical Ising spins $\{s_i\}$ placed on the N sites of the PC in any dimension $d \geq 2$ with ferromagnetic nearest-neighbor couplings J:

$$H = -J \sum_{\langle ij \rangle} s_i s_j. \tag{2}$$

The static critical behavior is known: Percolation clusters are "finitely ramified,"[2] so that $T_c = 0$. With a temperature variable $\theta = e^{-2J/T}$, the thermal correlation length[13,14] is given by

$$\xi_T(T) \sim e^{2\nu_p J/T} a \sim \theta^{-\nu_p} a, \tag{3}$$

where ν_p is the percolation correlation-length exponent and a is the lattice constant.

I assume a single-spin-flip (hence, spin-nonconserving) dynamics satisfying the usual detailed balance condition, e.g., Glauber dynamics[15]: spins flip at a rate $\sim e^{-\Delta E/T}\tau_0^{-1}$ (energy increased by ΔE) or $\sim \tau_0^{-1}$ (energy decreased or unchanged by flip). Then $\tau(L,T)$ is defined as the *typical* (median-among-realizations) lifetime of the slowest relaxing mode of a piece $\mathscr{P}$ of PC with diameter L.[16] The characteristic time of the infinite system is $\tau(T) = \lim_{L\to\infty}\tau(L,T) \sim \tau(\xi_T, T)$.

The first task is to derive Eq. (1). Take $L \ll \xi(T)$. It follows that $\mathscr{P}$ spends most of its time in its two ferromagnetic ground states with occasional transitions over the barrier separating them. Thus

$$\tau(\mathscr{P}, T) \sim e^{E_{\max}(\mathscr{P})/T}\tau_0, \tag{4}$$

where $E_{\max}(\mathscr{P})$ is an energy barrier—the highest energy on the lowest path connecting the ground states of $\mathscr{P}$ in its configuration space. The entropy term in the exponent of Eq. (4) can be neglected, compared to $E_{\max}/T$, as long as $L \ll \xi_T$.

To evaluate $E_{\max}(\mathscr{P})$, I will use the links-and-blobs model of PC's[17] (see Fig. 1). The PC can be pictured as a (contorted) chain of one-dimensional "links," interrupted by "decorations": multiply connected "blobs" in the chain, or "dangling ends" extending from it. The dangling ends and the blobs are made up of similarly decorated chains in a self-similar fashion.

Overturning $\mathscr{P}$ corresponds to moving a domain wall along the chain. On sections of purely one-

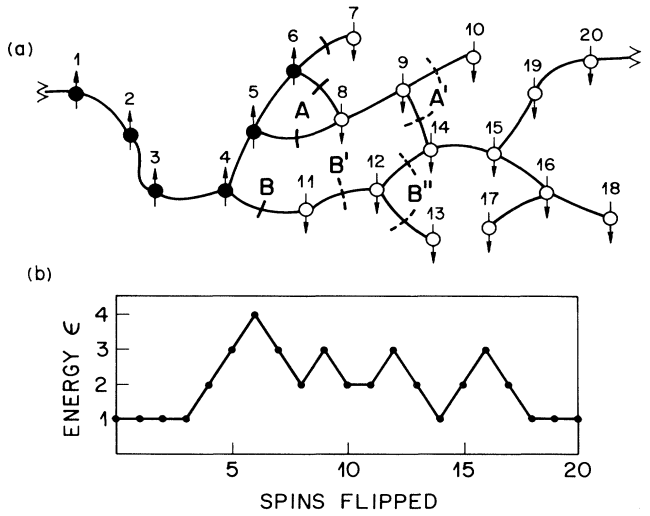

FIG. 1. (a) Schematic representation of part of a percolation cluster, showing one-dimensional links, dangling ends, and a blob with substructure. A domain of up-spin is spreading from the left. The spins are numbered according to an optimal sequence for flipping of the cluster. The bonds "violated" at the "worst" configuration in this sequence are shown slashed. (b) The energy as a function of the number of spins flipped.

dimensional chain (e.g., spins 1–3 in Fig. 1), the wall's energy is independent of its position, and so it does a random walk[18] with a rapid step rate $\sim \tau_0^{-1}$. However, moving the domain wall past a site where $n > 2$ bonds intersect (e.g., spin 4 or 15) requires "violating" $(n-2)$ additional bonds. Therefore, although the *static* ordering depends on the links,[14] the *dynamics* is dominated by barriers due to the decorations. One decoration can be completely overturned before the wall reaches the next one, and so $\epsilon_{\max}$ is the maximum of the independent barriers of the decorations.

What happens to the barrier when length is rescaled? Consider a blob $\mathscr{B}$ composed of two subchains in parallel (see Fig. 1); a domain wall moving across $\mathscr{B}$ need not advance along both subchains simultaneously, but only one at a time. The overall "worst" moment in the optimal sequence is when the wall cuts a subchain at the "worst" point of its "worst" subdecoration $\mathscr{B}'$ (at A in Fig. 1) while cutting the other subchain at one of its one-dimensional links (such as B in Fig. 1). Thus

$$\epsilon_{\max}(\mathscr{B}) = \epsilon_{\max}(\mathscr{B}') + 1 \tag{5}$$

(measuring energies in units of "violated" bonds, $\epsilon = E/2J$). In (5), $\mathscr{B}'$ is the overall "worst" of the subblobs from *both* subchains. Note the implicit assumption that, of the many optical flip sequences, there is at least one in which each spin, once flipped, never flips back. This property has been proved for the Sierpiński gasket[11] and for PC's on a Bethe lattice with threefold coordination,[12] and I conjecture that it is true for arbitrary clusters. The other optimal sequences are elaborations of such a sequence by temporary backtracking, side excursions, or alternative routes around states in the noncritical part $(E < E_{\max})$ of the sequence.

Let us make the scaling assumption that $E_{\max}(\mathscr{P})$ depends only on the diameter $L(\mathscr{P})$. Then $\mathscr{B}'$ is the subblob with the largest diameter L'. The Herrmann-Stanley[19] subblob distribution law implies $L' \sim b^{-1}L$, where $b > 1$ is the typical ratio of diameters of a blob to its largest subblob. Substituting into (5) yields

$$\epsilon_{\max}(L) = 1 + \epsilon_{\max}(b^{-1}L), \tag{6}$$

the solution of which is (1) with

$$Z = (\ln b)^{-1}. \tag{7}$$

I have neglected the possiblity that $\mathscr{B}'$ was a dangling end, so that the above argument actually gives Z for the backbone of the PC.[19] In fact a rule similar to (4) governs dangling-end barriers,[12] and so (1) and (7) should still hold for the full PC, with b(full PC) $< b$(backbone).

The argument leading to (1) and (7) becomes exact on deterministic self-similar networks, which are generated by stages in which each link is replaced by a piece of decorated chain. In particular, $b = 3$ for the nonrandom "squig" (see Mandelbrot and Given,[20] Fig. 2) which models the full PC in $d = 2$ ($b = 3$ for the squig backbone, too) and $b = 2^{4/3}$ on a hierarchical lattice modeling the PC backbone in $d = 2$.[21]

Furthermore, Eq. (1) is clearly a consequence of hierarchical geometry and hence ought to hold on any finitely ramified fractal lattice. It has been *proved* that (1) holds for the Sierpiński gasket,[11] with $Z = 1/2\ln 2$, and for PC's on Bethe lattices,[12] with $Z = 2/\ln 2$. (Note that these lattices are, respectively, all blobs and all dangling ends.) In fact PC's in dimension $d \geq 6$ have the same geometry as PC's on Bethe lattices,[12] and so (1) is essentially proved for PC's for $d \geq 6$.

The parameter Z for PC's is *universal* in that it depends only on d. I emphasize that Z is *not* in general a function of the Hausdorff (fractal)[1] or spectral (fraction) dimensions,[3,4] or of the percolation exponents; it seems to be yet another of the many geometrical parameters needed to characterize a fractal.[1]

I have ignored the contribution of very rare but very slowly relaxing compact clusters—regions which happen to be undiluted because of statistical fluctuations—which give rise to a "Griffiths phase" below the T_c of the *pure* system.[22] Consider the probability distribution of barriers $\epsilon_{\max}(\mathscr{P})$ in the ensemble of $\mathscr{P}$'s with diameter L. As a result of the compact clusters it has a tail $\sim L^d \exp(-c\epsilon_{\max}^{d/(d-1)})$ extending far beyond $\epsilon_{\max}(L)$. The largest such barrier in a *typical* domain of diameter L goes as $\epsilon_{\max}^{(\text{undiluted})} \sim C(d)$

$\times (\ln L)^{d/(d-1)}$. Thus, at lengths shorter than a crossover length $L^*(d)$, this tail dominates the barriers due to hierarchical geometry, and (1) fails.

I now turn to the consequences of (1) for the low-temperature dynamics. Substituting (1) into (4) gives

$$\tau(L,T) \sim (e^C L)^{2ZJ/T}\tau_0. \tag{8}$$

Thus $\tau(L,T) \sim L^{\tilde{z}(T)}$, where $\tilde{z}(T)$ diverges as $T \to 0$, in contrast to standard dynamic scaling.[7]

Equation (8) is valid only for $L \ll \xi_T$. To get $\tau(L \sim \xi_T, T) \sim \tau(T)$, it is necessary to include finite-temperature corrections, i.e., entropy effects. The most important of these turns out to be the existence of many alternative paths with the same, optimal barrier (e.g., configuration B′ instead of B in Fig. 1), so that relaxation is faster than the naive substitution $L \to \xi_T$ in (8).

Scaling can give the form of this correction. The static thermodynamic functions scale with ξ_T and PC's are self-similar (at $L \gtrsim a$), so that

$$\tau(L,T) = F(L/\xi_T(T))\tau(T), \tag{9}$$

with $\tau(a,T) \approx \tau_0$. Now Eq. (8) can be rewritten with use of (3):

$$\ln\tau(L,T) \approx Z\nu_p^{-1}\ln\xi_T \ln L - CZ\ln\xi_T \tag{10}$$

for small $\ln L$. (I am now setting $\tau_0 = a = 1$.) This can be reconciled with the form (9) only if there is an additional term of $O(\ln L)^2$ in (10), with

$$\ln F(x) \approx -\tfrac{1}{2}\nu_p^{-1}Z(\ln x)^2 + O(\ln x) \tag{11}$$

and

$$\begin{aligned}\ln\tau(T) &\approx \tfrac{1}{2}\nu_p^{-1}Z(\ln\xi_T)^2 + O(\ln\xi_T) \\ &\approx 2\nu_p Z(J/T)^2 + O(J/T) + \ldots;\end{aligned} \tag{12}$$

so finally

$$\begin{aligned}\ln\tau(L,T) \cong Z[-\tfrac{1}{2}\nu_p^{-1}(\ln L)^2 + (2J/T)\ln L] \\ + O(J/T, \ln L).\end{aligned} \tag{13}$$

The result (13) is contrasted with ordinary dynamic scaling[7] in Fig. 2. It is valid for $L \lesssim \xi_T$; when $L \sim \xi_T$ additional entropy corrections to (13) appear.

These results for $\tau(L,T)$ can be better understood by considering the nature of the dynamic renormalization group (RG)[7,23] which would give rise to them. Assume that we have a one-parameter, discrete static RG. Rescaling length by the natural rescale factor, b, we have $\xi_T' = \xi_T/b$, $\theta' = f(\theta) \cong b^{1/\nu_p}\theta$, with $f(0) = 0$ as the relevant fixed point. The simplest way to extend this to a dynamic RG has one time parameter τ. To give (9), its rescaling must have the form

$$\tau' = \lambda(\theta)\tau. \tag{14}$$

This is consistent with (12) only if

$$\lambda(\theta) \sim \theta \tag{15}$$

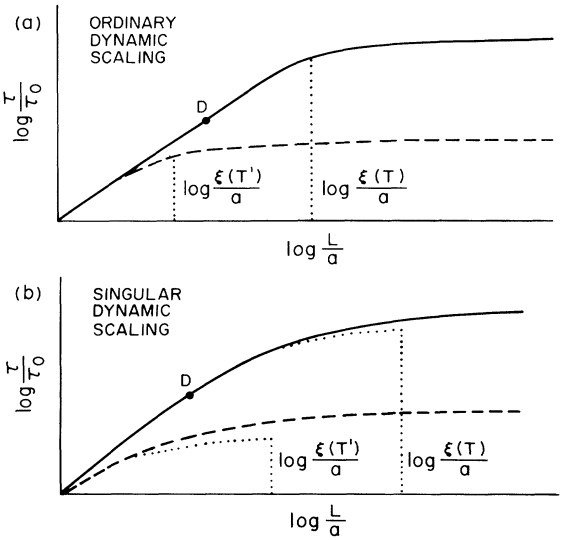

FIG. 2. Relaxation time as a function of length scale, for a very low temperature T (solid curve) and a higher temperature T' (dashed curve), for (a) standard dynamic scaling, and (b) singular dynamic scaling. In (b), the first terms in (13) give parabolas (dotted) which become level at $L \sim \xi_T$; with the corrections (solid, dashed) they level off only as $L \to \infty$. If the solid curve is translated so that point D is moved to the origin, it coincides with the dashed curve; this is the operation of the dynamic renormalization group.

for small θ. Ordinary dynamic scaling is also characterized by an RG equation like (14) except that in place of (15), ordinarily $\lambda(\theta) \to b^{-z}$, a constant, as $\theta \to 0$. Hence I propose the term "singular dynamical scaling" for the behavior (15). Note that if an approximate RG is constructed[10] which violates (15), it cannot give the correct behavior in the $T \to 0$ limit.

A somewhat different dynamic RG has been constructed for the Sierpiński gasket.[11] This renormalization group uses a master-equation approach: The dynamics is formulated as a random walk (with weights for each step) in configuration space, a diffusion problem which is equivalent to a resistor network. The network is self-similar and can be solved exactly, provided that *only* optimal paths are considered. This allows exact calculation of some nontrivial entropy corrections, which are similar in outline to those found here, but different in detail since the *static* RG is singular in this case[2] with $\theta' = \theta + O(\theta^2)$.

Other random systems with $T_c = 0$ have a dynamics described by energy-barrier activation (4), but of course singular dynamic scaling follows only if the barriers scale logarithmically. McMillan's phenomenological dynamic RG for an Ising spin-glass at its lower critical dimension (l.c.d.)[24] exhibits singular dynamic scaling, with a recursion like (15) leading to $\ln\xi_T \sim T^{-2}$ and $\ln\tau \sim T^{-3}$. On the other hand, in the random-field Ising model[25] at $d = 2$, the lower critical

dimension, the barriers scale as a power of L.

What does (12) imply for experiments? One signature of singular dynamic scaling is upward curvature in a $(\ln\xi_T, \ln\tau)$ plot. In a small temperature range about a given T, the data will be consistent with the effective exponent $\tilde{z} = d\ln\tau/d\ln\xi_T \sim T^{-1}$; this suggests why the neutron experiment[6] should find an anomalously large value for z. Actually the temperature range of Ref. 6 should be large enough to see curvature, but it does not. This is easily rationalized since ξ_T was still only $\sim 3a$ at the lowest T studied.

If simulations, or a repeat of the experiment at lower T, can reach the scaling regime ($\xi_T > L^*$), the curvature in (12) should be visible. Also, inelastic neutron scattering as a function of momentum transfer q could test (13). The best numerical way to check Eq. (1) would be exact evaluation of $\epsilon_{\max}$ for randomly generated PC's.

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