Relation between the Chiral Anomaly and the Quantized Hall Effect

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(Received 28 January 1985)

A simple method is used to calculate the currents induced by static uniform electromagnetic fields in the ground state of a two-dimensional electron gas. This is done for both the relativistic and the nonrelativistic cases. This analysis is used to explain the apparent similarity between the anomalous vacuum current in three-dimensional quantum electrodynamics and the quantized Hall current. The different natures of the two currents are demonstrated, and it is shown that the effective action for the gauge field in the nonrelativistic case does not contain a genuine Chern-Simons term. This precludes any attempt to use the chiral anomaly to explain the quantized Hall effect.

PACS numbers: 05.30.Fk, 11.30.Rd, 72.20.My

The experimental discovery that the Hall conductivity of a two-dimensional system is quantized in units of $e^2/2\pi\hbar$ [1] has led to extensive theoretical investigations.[2] One of the interesting ideas suggested[3] is the possible relation between the quantized Hall effect and the recently discovered chiral anomaly in (2+1)-dimensional gauge theories.[4] In 2+1 dimensions, a parity transformation corresponds to changing the sign of one of the two spatial coordinates, and, thus defined, parity is a symmetry of the classical action only if the fermions are massless. In the quantum theory, however, this symmetry is anomalous as one can see by examining the effective gauge-field action after doing the functional integration over the fermionic fields. This corresponds to the fact that external gauge fields drive currents of abnormal parity in the fermionic ground state. For quantum electrodynamics in 2+1 dimensions, the relation between the current and the external field appears to be similar to that found in the quantized Hall system, and corresponds to a Hall conductivity equal to $e^2/2\pi\hbar$ when the fact that the spin of the electron can take one of two values is taken into account. This analogy has been further extended[5] to explain the fractionally quantized Hall effect discovered later.[6] The motivation for these ideas is the hope that there may be an approximation in which the states near the Fermi surface can be described by a 2+1 massless Dirac theory in the same way this is possible, say, for some linear polymer systems.[7]

It is the purpose of this paper to settle this question. It will be shown that the two effects are of quite different natures, and that this is related to the physical definition of the current in each case, and not to the particular wave equation satisfied by the corresponding fields. The method I shall use will be to describe the two effects in the same simple language that makes clear the reason for their apparent similarity and less apparent differences.

I shall consider the motion of a noninteracting electron gas in the x-y plane under the effect of a uniform static magnetic field in the z direction, and a uniform static electric field in the plane of motion. The electric field can be chosen in the x direction without loss of generality. The gauge can be chosen such that

$$A_0 = -Ex, \quad A_x = 0, \quad A_y = Bx. \tag{1}$$

For simplicity, I shall use the natural units ($\hbar = c = 1$).

Let us consider the relativistic system. The Dirac matrices will be chosen to be

$$\gamma_1 = i\sigma_2, \quad \gamma_2 = -i\sigma_1, \quad \gamma_0 = \sigma_3.$$

It is convenient to distinguish between the two cases $eB > 0$ and $eB < 0$. In the first case, the normalized positive- and negative-energy wave functions for a particle of charge e and mass m are

$$u_{k,n}(x,y;t) = (4\pi\alpha_n)^{-1/2} e^{iky}\{(\alpha_n + m)^{1/2}\psi_n[x - x_0^+(k)]S_1 + (\alpha_n - m)^{1/2}\psi_{n-1}[x - x_0^+(k)]S_2\}\exp(-iw_{k,n}^+ t), \tag{2a}$$

$$v_{k,n}(x,y;t) = (4\pi\alpha_n)^{-1/2} e^{iky}\{(\alpha_n - m)^{1/2}\psi_n[x - x_0^-(k)]S_1 - (\alpha_n + m)^{1/2}\psi_{n-1}[x - x_0^-(k)]S_2\}\exp(-iw_{k,n}^- t). \tag{2b}$$

$\psi_n(x)$ is the nth normalized wave function of a harmonic oscillator of frequency Ω given by

$$\Omega = 2e(B^2 - E^2)^{1/2},$$

while k is a real number. S_1 and S_2 are two-component spinors given by

$$S_1 = \begin{pmatrix} i\cos\theta/2 \\ \sin\theta/2 \end{pmatrix}, \quad S_2 = \begin{pmatrix} i\sin\theta/2 \\ \cos\theta/2 \end{pmatrix}, \tag{2c}$$

where $\sin\theta = E/B$.

$$\alpha_n = [m^2 + 2eBn\cos\theta]^{1/2}, \tag{2d}$$

$$x_0^\pm(k) = (eB)^{-1}(k \pm \alpha_n/\cos\theta), \tag{2e}$$

$$w_{k,n}^\pm = -k\sin\theta \pm \alpha_n\cos\theta. \tag{2f}$$

For $eB < 0$, we can use the charge-conjugate wave functions

$$u'_{k,n} = \sigma_1 v^*_{-k,n}, \quad v'_{k,n} = \sigma_1 u^*_{-k,n} \tag{3}$$

with $\sin\theta$ replaced by $(-\sin\theta)$. It should be noted that $v_{k,0}$ and $u_{k,0}$ vanish for $m > 0$ and $m < 0$, respectively. (See Fig. 1.)

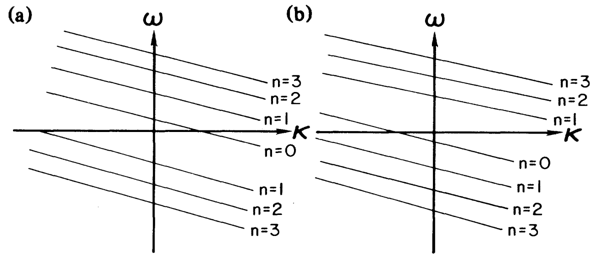

FIG. 1. The spectrum of a two-dimensional relativistic electron in a uniform static electric field in the plane of motion and a uniform static magnetic field perpendicular to it for $eB > 0$ [Eq. (2f)] when (a) $m > 0$, (b) $m < 0$.

The field operator can be written as

$$\psi(x,y;t) = \sum_{n=0}^{\infty}\int_{-\infty}^{\infty} dk\,[u_{k,n}(x,y;t)a_{k,n} + v_{k,n}(x,y;t)b_{k,n}^\dagger]. \tag{4}$$

The vacuum state $|0\rangle$ is defined by $a_{k,n}|0\rangle = 0$, $b_{k,n}|0\rangle = 0$ for all n.

The current J^μ is defined to be odd under the charge-conjugation symmetry

$$\psi(x,y;t) \rightarrow \sigma_1\psi^*(x,y;t).$$

We can easily check that this is true for

$$J_\mu(x,y;t) = -\tfrac{1}{2}e(\gamma_\mu)_{\beta\alpha}[\psi_\alpha(x,y;t), \bar\psi_\beta(x,y;t)]. \tag{5}$$

The vacuum current is given by

$$\langle 0|J_\mu|0\rangle = \tfrac{1}{2}e\int_{-\infty}^{\infty} dk\,(\bar v_{k,n}\gamma_\mu v_{k,n} - \bar u_{k,n}\gamma_\mu u_{k,n}). \tag{6}$$

Using the set of equations (6) for the eigenfunctions, one can evaluate the contribution to the current from one Landau level,

$$j_\mu = \int_{-\infty}^{\infty} dk\,\bar u_{k,n}\gamma_\mu u_{k,n} = \int_{-\infty}^{\infty} v_{k,n}\gamma_\mu v_{k,n},$$

which gives

$$j_0 = \frac{e|eB|}{2\pi}, \quad j_x = 0, \quad j_y = \frac{e^2E}{2\pi}\frac{eB}{|eB|}. \tag{7}$$

The vacuum current then reduces to

$$\langle 0|J_\mu|0\rangle = \tfrac{1}{2}j_\mu(N_- - N_+), \tag{8}$$

where N_+ and N_- are the numbers of positive- and negative-energy Landau levels, respectively. Note that this expression is not well defined for massless fermions because there is a zero-energy Landau level in this case. For $m \neq 0$, one can easily see that

$$N_+ - N_- = \frac{m}{|m|}\frac{eB}{|eB|}. \tag{9}$$

Equations (7)–(9) can be combined to give

$$\langle 0|J_0|0\rangle = -\frac{m}{|m|}\frac{e^2B}{4\pi}, \quad \langle 0|J_x|0\rangle = 0,$$
$$\langle 0|J_y|0\rangle = -\frac{m}{|m|}\frac{e^2E}{4\pi}, \tag{10a}$$

which can be written in the covariant form

$$\langle 0|J^\mu|0\rangle = -\frac{e^2}{8\pi}\frac{m}{|m|}\epsilon^{\mu\nu\alpha}F_{\nu\alpha}. \tag{10b}$$

Equation (10) clearly violates parity conservation since it tells us that the parity-even current components J_0 and J_y are proportional to the parity-odd external fields B and E_x.

Consider next the case of the quantized Hall effect in a nonrelativistic two-dimensional electron gas. The field in this case satisfies the Schrödinger equation. The solutions of the time-independent Schrödinger equation are

$$\psi_{k,n,\sigma} = (2\pi)^{-1/2}e^{iky}\psi_n(x)S(\sigma) \tag{11a}$$

corresponding to the energies (for $eB > 0$)

$$w(k,n,\sigma) = (n + \tfrac{1}{2} - \sigma)\frac{eB}{m} - k\sin\theta - \frac{m}{2}\sin^2\theta, \tag{11b}$$

where $\psi_n(x)$ is the normalized nth wave function of a harmonic oscillator of frequency em/B, while $\sigma = \pm\frac{1}{2}$ is the z component of the spin of the electron, and $s(\sigma)$ is the spin wave function. (See Fig. 2.)

The current carried by one Landau level is given by

$$j_0 = e\int_{-\infty}^{\infty} d\kappa\,|\psi_{n,k,\sigma}|^2 = e|eB|/2\pi, \tag{12a}$$

$$j_x = e\int_{-\infty}^{\infty}\psi^*_{n,k,\sigma}\left(-\frac{i}{2m}\overleftrightarrow{\partial}_x\right)\psi_{k,n,\sigma} = 0, \tag{12b}$$

$$j_y = e\int_{-\infty}^{\infty}\psi_{n,k,\sigma}\left(-\frac{1}{2m}\overleftrightarrow{\partial}_y - \frac{e}{m}A_y\right)\psi_{k,n,\sigma} = \frac{e^2E}{2\pi}\frac{eB}{|eB|}, \tag{12c}$$

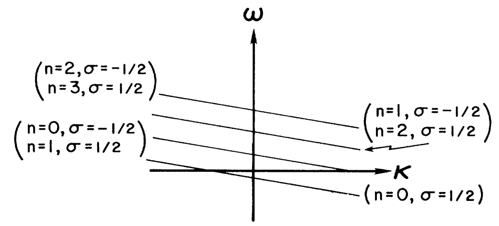

FIG. 2. The spectrum of a two-dimensional electron moving under the action of the same fields of Fig. 1 [Eq. (11b)].

exactly as in the relativistic case. But, unlike the relativistic case, the current is defined here to be the current carried by all the Landau levels below the Fermi energy. Let there be n such levels. Then we have

$$\rho = J_0 = n\,(e^2B/2\pi)\,\mathrm{sgn}(eB), \tag{13a}$$

$$J_y = n\,(e^2E/2\pi)\,\mathrm{sgn}(eB). \tag{13b}$$

The above relations differ in two ways from those in the relativistic case: There is an extra factor $2n$, and more importantly, there is no violation of parity conservation since both sides of both equations are parity even. The effective Lagrangian of the gauge field does not contain a genuine Chern-Simons term, but something similar that contains the extra factor $\mathrm{sgn}(eB)$, at least for uniform static fields, and a generalization thereof for arbitrarily varying fields. The matters are further complicated by the fact that, when B vanishes, the above-described picture of the Landau levels is no longer valid, and the dissipative effects become crucial.

At the risk of repeating myself, I shall state that this difference is a consequence of the definition of the currents in the two cases, and not the type of wave equation satisfied by the quantum field. It is probably not very useful for the understanding of the quantized Hall effect to make use of the Dirac theory.

The following final remarks may be in order. If we consider a (2+1)-dimensional relativistic system of finite electron density, the ground-state current will be the sum of the parity-nonconserving current given by Eq. (10), and a parity-conserving piece given by Eq. (13). The effective gauge-field Lagrangian will have a genuine Chern-Simons piece,[3] and a "pseudo–Chern-Simons" term having a factor of $\mathrm{sgn}(eB)$, or its generalization for arbitrarily varying fields. The pseudo–Chern-Simons term breaks charge-conjugation symmetry. This is understandable since there is no such symmetry to begin with in the Schrödinger theory, while the finite-density state in the Dirac theory is not the vacuum state, and not an eigenstate of the charge-conjugation operator.

It is important to remember that the problem of the quantized Hall effect is not to calculate the current in the free-electron approximation, but to explain why this current survives the interactions of the electrons with the imperfect lattice and with each other. An anomalous current would not be affected by such considerations, and this is why relating the quantized Hall effect to the anomaly had seemed at first to be such an attractive idea. Since, as I have shown, there is no actual anomaly in the system, it is probably quite useless to pursue this idea any further.

It has been recently suggested[8] that the vacuum state of three-dimensional QED should be treated as a collective state similar to the ground state of Laughlin's theory of the fractional Hall effect[9] in which the Coulomb interactions between the electrons in the same Landau level are taken into account. I do not think that this is true because the effect of the Coulomb interaction is quite different in the two cases, not because the Coulomb potential is logarithmic in three-dimensional QED and inversely proportional to the distance in the case of the Hall effect, a fact that Laughlin has pointed out to be of secondary importance, but because definition of the charge density in QED to be odd under charge conjugation leads to a Coulomb interaction that couples both filled and empty states. Then the electrons in the zero-energy Landau level see a positive background of charge density equal to that carried by one half-filled Landau level. This is quite different from the neutralizing background in the Hall system. The problem in the latter system is not to find the filling factor of the Landau level that minimizes the energy, but to find the values of the filling factor for which there are collective Laughlin states.

I have benefitted from conversations with G. Baskaran, J. Distler, and R. Wijewarhana, to whom I would like to express my thanks. This work was supported in part by the National Science Foundation under Grant No. PHY-80-19754.

[1]K. V. Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. **45**, 494 (1980).

[2]R. B. Laughlin, Phys. Rev. B **23**, 5632 (1981); B. I. Halperin, Phys. Rev. B **25**, 2185 (1982).

[3]R. Jackiw, Phys. Rev. D **29**, 2375 (1984), and Comments Nucl. Part. Phys. **13–15**, 141 (1984); K. Ishikawa, Phys. Rev. Lett. **53**, 1615 (1984).

[4]N. Redlich, Phys. Rev. Lett. **52**, 18 (1984); L. Alvarez-Gaumé and E. Witten, Nucl. Phys. **B234**, 269 (1984).

[5]A. Friedman *et al.*, Phys. Rev. Lett. **52**, 1587 (1984), and Northeastern University Report No. NUB-2662, 1984 (to be published).

[6]D. C. Tsui, H. L. Stormer, and A. C. Gossard, Phys. Rev. Lett. **48**, 1559 (1982), and **50**, 1953 (1983).

[7]W. Su, J. Schrieffer, and A. Heeger, Phys. Rev. Lett. **42**, 1698 (1979).

[8]K. Johnson, to be published.

[9]R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).