Symmetry Criterion for the Lack of a Stable Fixed Point in the Renormalization-Group Recursion Relations

Louis Michel
Institut des Hautes Etudes Scientifiques, F-91440 Bures sur Yvette, France

and

Jean-Claude Toledano
Centre National d'Etudes des Télécommunications, F-92220 Bagneux, France
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Within the framework of the ϵ expansion, we establish a symmetry criterion allowing us to predict if a phase transition is driven first order by the fluctuations. It is expressed in the form of group-theoretical conditions imposed on the Hamiltonian symmetry. In particular, we specify a sufficient condition for the lack of a stable-fixed-point Hamiltonian. The effectiveness of this condition is illustrated by its application to Hamiltonians based on four-component order parameters.

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In the framework of Landau's theory of phase transitions,[1] the well-known "symmetric-cube rule"[1] provides a sufficient group-theoretical condition imposed on the order parameter for the occurrence of a first-order transition. Since 1976 several authors[2–5] have suggested that similarly, in the framework of the renormalization-group (RG) method, a group-theoretical criterion should, in certain cases, preside over the occurrence of a fluctuation-induced first-order character. These authors have referred to the situation where this character arises from the absence of a stable fixed point (FP) in the flow of Hamiltonians considered in the RG method in reciprocal space. Indeed, the lack of a stable FP results from the form of the fourth-degree terms in the Hamiltonian which is entirely determined by the symmetry properties of the order parameter. Thus one could hope to detect the lack of a stable FP on the sole basis of the former symmetry properties and avoid, in certain cases, the actual solving of the RG recursion relations.

Up to now the specific group-theoretical conditions defining the Hamiltonians lacking a stable FP had not been clarified. This problem is essentially of interest for phase transitions whose order-parameter dimension n is larger than three since for $n \leqslant 3$ a stable FP always exists.[6] Two previous attempts in this direction are worth considering.

In the first place, it had been stated that the relevant symmetry indicator could be the number of linearly independent fourth-degree terms contained in the Hamiltonian. More precisely, there would be no stable FP if there were more than a small number (three or four) of independent terms.[4,5] Recent investigations by Michel[7] and by Grinstein and Mukamel[5] have weakened this conjecture. These authors have displayed examples of Hamiltonians containing an arbitrary number of fourth-degree terms, and nevertheless possessing a stable FP, within the ϵ expansion.

On the other hand, Korzhenevskii[3] has examined the connection between the characteristics of the stable FP and the group of covariance of the Hamiltonian (which coincides with the normalizer defined hereunder). This author has pointed out, in particular, that for "low-symmetry" Hamiltonians involving many fourth-degree terms, the stable FP, if it exists, should always display an increase of symmetry. However, no explicit rule was stated for the absence of a stable FP.

In this Letter, extending the arguments developed by Korzhenevskii, we are able to establish two group-theoretical results. *The first result specifies the possible symmetries of stable fixed points. The other provides for the first time, a sufficient condition for the lack of a stable fixed point.* These two results rely on the uniqueness of the stable FP which has recently been established[8,9] within the two-loop order ϵ expansion.

The effective Hamiltonian density to consider is

$$H(\mathbf{x}) = \left(-\frac{r}{2}\right)\left[\sum_1^n \phi_i^2(\mathbf{x})\right] + \sum_1^n (\nabla\phi_i)^2 + \sum_{\nu=1}^p u_\nu O_\nu(\phi_i), \tag{1}$$

where the n functions $\phi_i(\mathbf{x})$ are the local values of an n-dimensional irreducible order parameter. The last term of Eq. (1), which we denote P_4, is a linear combination, with arbitrary coefficients u_ν, of p homogeneous polynomials $O_\nu(\phi_i)$ of degree four, linearly independent in the vector space generated by the $\binom{n+3}{4}$ monomials of the ϕ_i components.[7] Each O_ν is invariant under the symmetry group of the high-symmetry phase adjacent to the transition. P_4 is the most general homogeneous polynomial of degree four invariant under the former group.

The RG method associates to P_4 a flow of polynomials depending on the same invariants O_ν, and having continuously varying u_ν values. Physically, all the Hamiltonians relative to a given trajectory of the flow correspond to the same critical behavior. The universality of the critical behavior stems from the existence of a stable FP in the flow. The characteristics of the flow are determined by the recursion relations

$$du_\nu/d(\log\lambda) = \beta_\nu(u_{\nu'}), \quad (2)$$

where λ parametrizes the trajectories (the critical singularities correspond to $\lambda \to 0$) and where the expression of the β_ν functions is known in the framework of the ϵ expansion.[6] A FP u_ν^* satisfies the set of equations $\beta_\nu(u_{\nu'}^*) = 0$, while a stable FP is defined, in addition, by the positiveness of the real part of the eigenvalues of the matrix $(\partial\beta_\nu/\partial u_{\nu'})|_{u_\nu^*}$.

In order to examine the invariance properties of P_4 and of the associated flow, the symmetry transformations of interest[9,10] are the linear orthogonal transformations acting in the n-dimensional order-parameter space carried by the ϕ_i. Three symmetry groups, subgroups of $O(n)$, can be defined in relation with the "full symmetry" of P_4 [which is also the symmetry of $H(\mathbf{x})$].

The centralizer G_c of P_4 is the largest subgroup of $O(n)$ leaving invariant every polynomial $(\sum u_\nu O_\nu)$ with arbitrary u_ν coefficients. In other terms, G_c leaves invariant every vector of the p-dimensional space E spanned by the O_ν. On the other hand, each given vector in E, i.e., each given polynomial with specified coefficients u_ν^0, has a full symmetry group G_0 which is called its little group. Clearly G_c is the intersection of all the G_0 ($G_c \subseteq G_0$). A group G_0 can be regared as the centralizer of a subspace of E (possibly reduced to the sole direction considered). Finally, the normalizer G_N of E is the largest subgroup of $O(n)$ preserving E as a whole. It transforms any polynomial P_4 into a combination of the same O_ν but with generally different coefficients u_ν. G_c is an invariant subgroup of G_N ($G_N \supseteq G_c$). Consistently, Korzhenevskii[3] had labeled G_N the "covariance group" of the Hamiltonian $H(\mathbf{x})$.

The RG-recursion relations defined by Eq. (2) have an important symmetry property which has been recognized by a number of authors.[3,11,12] They are covariant under the transformations of $O(n)$. This property expresses the physical requirement that they do not decrease the symmetry of the system considered. More precisely, the little group G_0 of P_4 is strictly preserved along a trajectory except, possibly, at a FP, where the symmetry can increase.[9,12] Thus, to each trajectory, a little group G_0 can be associated. On the other hand, in conformity with its definition, the centralizer G_c leaves invariant each point of every trajectory. Let us examine the action of the normalizer G_N.

As G_N does not change the form of P_4 in Eq. (1), it will leave invariant the set of trajectories composing the flow of the u_ν coefficients, as well as the pattern of fixed points. Each element of G_N will transform a given trajectory, characterized by the little group G_0, into a trajectory possessing a little group G_0' which is a conjugate of G_0 (i.e., $G_0' = SG_0S^{-1}$, where $S \in G_N$). The action of $S \in G_N$ can be considered as a mere change of the reference frame in the space of the ϕ_i components, and therefore, it establishes a correspondence between physically equivalent trajectories of the flow. Likewise, a FP is transformed by $S \in G_N$ into a physically equivalent FP possessing a conjugated symmetry group and the same stability, and describing the same critical behavior.[3,12]

It has been shown recently that at the lowest conclusive order in the ϵ expansion, if a stable FP exists in the RG flow, it is unique. For $n \neq 4$, this property could be proved[8] by means of the one-loop order ϵ expansion while for $n = 4$, it has been necessary to examine the two-loop order recursion relations.[9] This is a nontrivial result, since the uniqueness of the stable FP is not a physical requirement. A well defined critical behavior, which is indeed required, would be achieved even if several stable FP existed, provided that these FP were physically equivalent (i.e., symmetry related) in the manner described above.

This uniqueness imposes the condition that a stable FP cannot have symmetry equivalents, and accordingly it puts severe restrictions on its possible symmetries. Let u_ν^* be a FP in a certain polynomial space E having G_c as a centralizer. The symmetry of u_ν^* is defined as the little group G^* of the polynomial $\sum u_\nu^* O_\nu$. The group G^* is also the centralizer of a space E^*, containing u_ν^*, and contained in E. If u_ν^* is stable in E, it is also stable in the subspace E^*. From the uniqueness of the stable FP in E^*, we deduce that u_ν^* is necessarily invariant under the normalizer G_N^* of E^*. This requires $G_N^* \subseteq G^*$. The inclusion relation between the normalizer and the centralizer of E^* yields the converse relation $G^* \subseteq G_N^*$. Thus (a) *a stable fixed point is necessarily characterized by the coincidence of the centralizer and the normalizer associated to it:* $G^* = G_c^* = G_N^*$.

On the other hand, we can take into account the uniqueness of the stable FP in the space E. This imposes on the normalizer G_N of E the condition $G_N \subseteq G^*$. This condition can be used in two ways. If one considers a given FP, and finds that the action of G_N does not leave it invariant, then this FP cannot be stable. In a more general way, if we have determined beforehand for a given value of n the set of groups $G_i^* \subseteq O(n)$ complying with condition (a), we can formulate the following sufficient condition: (b) *If* G_N *is not a subgroup of at least one of the* G_i^*, *the considered* *flow in E has no stable fixed point.* Criteria (a) and (b) are well-defined symmetry conditions which can be worked out on the sole basis of the knowledge of the Hamiltonian symmetries.

In the first place, a list of the groups G_i^* has to be established for each order-parameter dimension n. In this view one can use a group-theoretical method described in Ref. 10 in order to select, up to a conjugation, all the centralizers of P_4 polynomials among the irreducible subgroups on $O(n)$. This method also provides, for each G_c, the explicit form of the invariant polynomials P_4. Furthermore, the knowledge of this explicit form permits the determination of the normalizer G_N by examining systematically the action, on P_4, of the groups containing G_c and selecting among them G_N. The enumeration of the G_i^* is achieved by an inspection of the preceding results in order to single out the polynomials for which $G_c = G_N$.

The application of condition (b) to a given Hamiltonian consists in the identification of its normalizer $G_N' \subseteq O(n)$, and the checking of its possible inclusion in the G_i^*.

It is easy to see that, for any value of n, one of the G_i^* coincides with $O(n)$: On the symmetry basis of condition (a), the "isotropic" FP is always a possible stable FP. From previous ϵ-expansion calculations,[6,8] we know that this FP is indeed stable in two situations: when $n \leqslant 3$, and when the Hamiltonian, with $n > 3$, has the $O(n)$ symmetry itself. In the remaining cases, i.e., for anisotropic Hamiltonians with $n > 3$, the isotropic FP is never stable,[6,8,9] and the two formulated conditions are expected to provide nontrivial results by considering, in their application, only the G_i^* which are strict subgroups of $O(n)$. Let us examine this application in the case $n = 4$.

In a recent work,[9] the centralizers G_c and the normalizers G_N of P_4 polynomials have systematically been determined for this value of n. Up to a change in coordinates in O(4), one finds 21 possible forms of anisotropic Hamiltonians associated to 21 centralizers $G_c \subset O(4)$. Only three of these symmetries comply with condition (a). They represent the possible symmetries G_i^* $(i = 1, 2, 3)$ of stable FP for $n = 4$. Their characteristics as subgroups of O(4) are detailed in Ref. 9 and will not be discussed here. We note, however, that the three corresponding E^* spaces have two dimensions. Also, the G_i^* comprise the "hypercubic" and the "dicylindrical" symmetries which are the symmetries of stable FP, determined by the ϵ expansion, in the examples of $(n = 4)$ Hamiltonians studied respectively by Aharony[13] and by Mukamel.[14]

As an illustration of the use of condition (b), consider the Hamiltonian associated to the polynomial with five indendent terms:

$$P_4 = u_1\left(\sum \phi_i^4\right) + u_2(\phi_1^2\phi_2^2 + \phi_3^2\phi_4^2) + u_3(\phi_1^2\phi_3^2 + \phi_2^2\phi_4^2) + u_4(\phi_1^2\phi_4^2 + \phi_2^2\phi_3^2) + u_5\phi_1\phi_2\phi_3\phi_4. \quad (3)$$

As shown in Ref. 9, the normalizer G_N of P_4 is a finite subgroup of O(4), or order 2304, which is not a subgroup of any of the three G_i^* mentioned above. The lack of a stable FP can thus be asserted for the corresponding Hamiltonian. This result is confirmed by the effective solving of the two-loop order fixed-point equations relative to it.[9]

More generally, condition (b) appears very restrictive since it allows us to show that among the 21 Hamiltonians of interest for $n = 4$, 10 lack a stable FP for symmetry reasons.[9] For the remaining 11, the solving of the FP equations discloses the absence of a stable FP in 6 additional Hamiltonians.

For higher values of n, less precise statements can be made because of the incompleteness of the information available at present on the irreducible subgroups of $O(n)$. An incomplete list of the G_i^* for various values of n has recently been worked out.[15] It can be noted, for instance, that for $n > 2$, the generalized cubic symmetry in n dimensions always satisfies condition (a): The cubic FP is always a possible stable FP, in agreement with the calculations of Aharony.[13] The cubic groups in n dimensions, $\Gamma_{1,n}$ are part of a set of groups denoted $\Gamma_{p,q}$ (with $pq = n$).[7] These groups are of the form $\Gamma_{p,q} = [O(p)]^q \times \Pi_q$, where $[O(p)]^q$ is the direct product of q orthogonal groups in p dimensions, and where Π_q realizes all the possible permutations of the q p-dimensional spaces. Each $\Gamma_{p,q}$ is[7] the centralizer of a two-dimensional space E of fourth-degree polynomials in $O(n)$. Besides, all the $\Gamma_{p,q}$ comply with condition (a) and are possible symmetries of stable FP. Actually, they represent the symmetries of one family of stable FP found in the set of examples displayed by Michel[7] of Hamiltonians with many fourth-degree terms, and also in the smaller set investigated by Grinstein and Mukamel.[5]

In the light of conditions (a) and (b) we can see that the previous inferences relating the lack of a stable FP to the number of terms in P_4, or assigning a high symmetry to the stable FP,[3-5] were, in fact, approximately correct. Indeed, a low-symmetry Hamiltonian will not often possess a stable fixed point because the normalizer G_N of its many-dimensional space E will generally (but not necessarily) be a high-symmetry subgroup on $O(n)$. However, while the earlier conjectures only provided a general trend, the criteria presented in the present paper define precise constraints.

Though these criteria are based on a property established at the lowest orders of the ϵ expansion (i.e., the uniqueness of the stable FP), their formulation in terms of the essential symmetries of the Hamiltonian suggests that they should hold independently of this approximation. In favor of this conjecture, we note that in one of the examples[16] with $n=4$ for which condition (b) allows us to assert the absence of a stable FP, Mukamel and Wallace[16] have confirmed this absence by means of nonperturbative arguments.

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