Flicker ($1/f$) Noise in Percolation Networks: A New Hierarchy of Exponents

R. Rammal
Centre de Recherche sur les Très Basses Températures, Centre National de la Recherche Scientifique, F-30842 Grenoble Cedex, France

and

C. Tannous
Département de Physique, Ecole Polytechnique, Montréal, Québec H3C 3A7, Canada

and

P. Breton and A.-M. S. Tremblay
Département de Physique et Centre de Recherche en Physique du Solide, Université de Sherbrooke, Sherbrooke, Québec J1K 2R1, Canada

(Received 10 December 1984)

New results for the magnitude of flicker noise, considered as resistance fluctuations, in random resistor networks are reported. Near the percolation threshold p_c, the magnitude of the relative noise is shown to diverge as $(p-p_c)^{-\kappa}$. The new exponent κ is calculated by various methods: Monte Carlo simulations, effective-medium theory, and position-space renormalization group. Exponents pertaining to higher-order cumulants of the resistance fluctuations are also calculated. The possible implications of our results for ongoing experiments on metal-insulator mixtures and cermets are also discussed.

PACS numbers: 72.70.+m, 05.40.+j

Flicker ($1/f$) noise[1] refers to the low-frequency spectrum of excess fluctuations measured when a constant current is applied to a resistor. That spectrum almost always has a power-law form $\omega^{-\alpha}$ with α close to unity. The origin of this power law has been the subject of innumerable controversies and is not the purpose of the present paper. Instead, we concentrate on the behavior of the magnitude of resistance noise in random resistor networks. Our work is based on two well-established properties of $1/f$ noise: (a) It is resistance noise[2] and (b) the resistance fluctuations are correlated over microscopic distances only.[3] Our results have direct implications for ongoing experiments on metal-insulator mixtures[4] and possibly Ni-Al_2O_3 cermets.[5] Namely, we predict that the magnitude of the noise diverges as $(p-p_c)^{-\kappa}$ when one approaches the percolation threshold p_c. Here κ is a new exponent which is a member of a new hierarchy of exponents describing the cumulants of the resistance fluctuations. Evidence[6] for the existence of such exponents has been already given in the case of self-similar structures.

The purpose of this Letter is to present our model and to report new results on the magnitude of the noise, obtained by Monte Carlo simulations, in the bond-percolation problem on a two-dimensional square lattice. Our results are compared to the predictions of effective-medium theory as well as to estimates of κ obtained with a position-space renormalization group. In addition to κ, our results confirm the existence of other exponents and are thus relevant to the physics of self-similar structures in general. This new hierarchy of exponents provides a means for the characterization of a given structure in its fine details with a large set of independent measurable quantities.

The simplest version of the model can be formulated as follows. The branch resistances, assumed to have the same value r, fluctuate independently in time on each bond with a correlation function whose Fourier transform is

$$\langle \delta r_\alpha(\omega)\delta r_\beta(-\omega)\rangle = \rho^2(\omega)\delta_{\alpha\beta}. \tag{1}$$

As long as each bond resistance fluctuates independently with the same spectrum, the explicit frequency dependence can be discarded. Actually, that spectrum is of the form $\rho^2(\omega)\sim\omega^{-\alpha}$. The magnitude of the relative noise $\mathcal{S}_R$ for the resistor network is found[1,6] from Tellegen's theorem:

$$\mathcal{S}_R \equiv \langle \delta R\, \delta R\rangle / R^2 = \mathcal{s}\,(\textstyle\sum_\alpha i_\alpha^4)/(\sum_\alpha i_\alpha^2)^2, \tag{2}$$

where R and δR are respectively the overall resistance and its time fluctuation while i_α is the current that flows in a branch α in the steady-state obtained without resistance fluctuations. Explicit frequency dependence of $\mathcal{S}_R$ enters via $\mathcal{s}\equiv\rho^2(\omega)/r^2$, i.e., the relative noise for an individual branch. By means of Eq. (2), lower and upper bounds for $\mathcal{S}_R$ may be found[6]:

$$1/N_b \le \mathcal{S}_R/\mathcal{s} \le r/R, \tag{3}$$

where N_b is the total number of conducting branches in the network. The lower bound is obviously reached on regular Euclidean networks: $\mathcal{S}_R \simeq \mathcal{s}\,L^{-d}$, where L

denotes the length scale of the system, of Euclidean dimension d. However, for a self-similar network, one can show that $\mathcal{S}_R/\mathcal{S} \simeq L^{-b}$, where b denotes a new exponent. In general, the exponent b is neither related to the fractal dimension $\bar{d}$ nor to the resistance exponent defined by $R(L) \simeq L^{-\beta_L}$. Using Eq. (3), one deduces that $-\beta_L \leq b \leq \bar{d}_B$, where $\bar{d}_B$ denotes the fractal dimension of the backbone. The exponent b has been calculated for different families of fractal structures.[6] In particular, for percolation clusters, one can show that $b=1$ at $d=1$ and $b=2$ for $d \geq 6$. In what follows we shall focus on bond-dilute networks, where bonds having a resistance r occur independently with probability p. The noise associated with the whole lattice, as a function of the filling fraction p, is one of the relevant quantities that we study.

The simulation method we use for square-lattice networks is a modification of the transfer-matrix method[7] generalized along the lines of Limieux, Breton, and Tremblay.[8] Resistances at the top of the network are grounded while resistances at the bottom are shunted and fed with a current equal to unity. One needs the voltages v_k at every node k in order to obtain the current in every branch α [Eq. (2)]: $i_\alpha = i_{kl} = g_\alpha(v_k - v_l)$. Here i_α is the current in branch α between nodes k and l and g_α is the conductance ($1/r$ or 0) of that branch. Recursion relations for every voltage node may be derived by the inclusion in the generating function of Ref. 8 of one source field at every mode. We improved the algorithm by isolating the backbone with use of the method of Herrmann, Hong, and Stanley.[9] For every lattice size, ranging from 10×10 to 23×23, about 3000 conducting samples were studied. All exponents quoted in the following page are obtained through standard finite-size scaling analysis[7]; i.e., calculations are done at p_c and exponents are obtained from fits to log-log plots of the size dependence of the averaged physical quantities. Samples which do not conduct are rejected.

For the square lattice our results are illustrated in Fig. 1, where $s(p)/\mathcal{S} = \mathcal{S}_R(p,L)/\mathcal{S}_R$ $(p=1,L)$ is the normalized relative noise, for a system of linear size L, plotted as a function of p. Far from $p_c=0.5$, the numerical results follow the effective-medium-theory (EMT) prediction (see below), $s(p)/\mathcal{S} = (2p-1)^{-1}$, to a high accuracy for any $p \gtrsim 0.65$. The expected deviations from EMT occur near p_c as usual. For a finite system, $s(p)/\mathcal{S}$ saturates at p_c, where the self-similarity of the conducting part enters into consideration. Close to p_c, we rely on finite-size scaling to analyze the data. We make the usual scaling hypothesis:

$$\mathcal{S}_R(p,L) = L^{-b} f(\xi_p/L), \tag{4}$$

where $\xi_p \simeq |p-p_c|^{-\nu_p}$ is the percolation length, and $f(x)$ denotes a scaling function describing the fractal-to-Euclidean crossover. In the fractal regime ($x \gg 1$) we expect $f(x)$ to be independent of x, i.e., $\mathcal{S}_R(p,L) \sim L^{-b}$ for $\xi_p \gg L$, whereas in the Euclidean one ($x \ll 1$) we must recover the L^{-d} size dependence, which implies that $f(x \ll 1) \sim x^u$ with $u = d-b$. Therefore, $\mathcal{S}_R(p,L) \simeq L^{-d}(p-p_c)^{-\nu_p(d-b)}$, which leads to $s(p) \sim (p-p_c)^{-\kappa}$ with $\kappa = \nu_p(d-b)$ as announced in the introduction.[6] The divergence of the noise close to p_c occurs as expected. The exponent b has been calculated, by means of the power-law behavior $\mathcal{S}_R(p_c,L) \sim L^{-b}$ at p_c (see Fig. 1). The value of b that we obtain is $b = 1.16 \pm 0.02$ which, with $\nu_p = \frac{4}{3}$, implies that $\kappa = 1.12 \pm 0.02$. Note that b lies within the theoretical bounds $-\beta_L \leq b \leq \bar{d}_B$ with $-\beta_L = 0.973 \pm 0.005$ and $\bar{d}_B = 1.62 \pm 0.02$ in two dimensions.[10] Bounds for κ are trivially deduced from the above. Note also that κ appears as an increasing function of d since it starts from $\kappa=1$ at $d=1$, is equal to 1.12 at $d=2$, and reaches the value $\kappa=2$ at $d \geq 6$.

Using the same numerical data, we have been able to calculate the first members of the hierarchy of exponents introduced in Ref. 6. That hierarchy is defined with the help of the geometrical factors which relate, within the independent-resistor model, higher-order cumulants of the overall resistance fluctuations to the cumulants of the individual resistance fluctuations. For example,

$$\langle \delta R^4 \rangle - 3\langle \delta R^2 \rangle^2 = G_4(\langle \delta r^4 \rangle - 3\langle \delta r^2 \rangle^2), \tag{5}$$

where $G_{2n}(n \geq 0)$ is defined by $G_{2n} = \Sigma_\alpha i_\alpha^{2n}$. G_{2n} is expected to scale in the fractal regime as $G_{2n} \sim L^{-x_n}$.

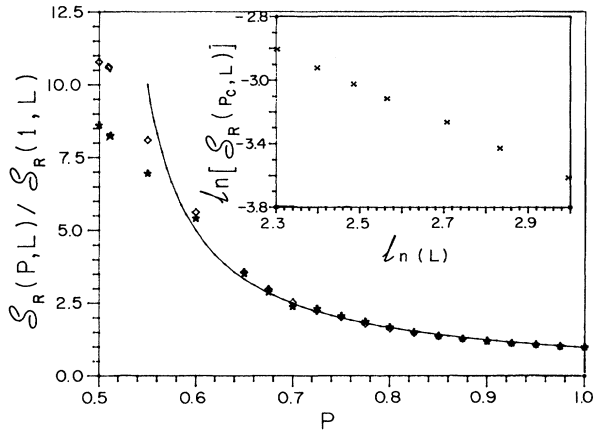

FIG. 1. Monte Carlo results for the relative noise $s(p)/\mathcal{S} = \mathcal{S}_R(p,L)/\mathcal{S}_R(1,L)$ for the bond-percolation problem on a two-dimensional square $L\times L$ lattice ($p_c=0.5$). Different symbols correspond to different L: asterisks, $L=15$; lozenges, $L=20$. The solid line represents the effective-medium-theory prediction $s_m(p)/\mathcal{S} = (2p-1)^{-1}$. The inset shows the power-law behavior of $\mathcal{S}_R(p_c,L)$ vs L at threshold.

For the bond-percolation model, we obtain

$$x_0 = -\bar{d}_B = -1.65 \pm 0.02,$$

$$x_1 = \beta_L = -t/\nu + (d-2) = -0.978 \pm 0.01,$$

$$x_2 = b + 2\beta_L = -0.81 \pm 0.02,$$

$$x_3 = -0.77 \pm 0.03, \quad x_4 = -0.74 \pm 0.02$$

consistent with the inequalities[6]

$$x_{n-1} \leq x_n \leq x_{n-1}[n/(n-1)] + \bar{d}_B(n-1)$$

($n > 1$ in the upper bound). These results confirm the fact that there are at least four independent exponents. Our values for x_0 and x_1 agree with those of the literature. Furthermore, the value of b extracted from x_1 and x_2 is consistent with the one previously quoted. Note that the quantities which are averaged over are different in these two ways of evaluation of b. The asymptotic result[11] $x_\infty = -\frac{3}{4} = -1/\nu$ can be understood from the scaling of singly connected bonds.[12]

Far from p_c, the behavior of the noise can be accounted for by the effective-medium theory. Such a theory, previously developed for the conductivity,[13] has been extended for the noise.[14] We give the results for the bond problem.

We find that the EMT value for the relative noise $s_m(p)$, normalized to σ at $p = 1$, is

$$s_m(p)/\sigma = \begin{cases} \dfrac{z-2}{z}(p-p_c)^{-1}, & p > p_c, \\ 0, & p < p_c, \end{cases} \tag{6}$$

where $p_c = 2/z$ and z is the lattice coordination number. Note that for the particular case considered here, $g_m(p)\, s_m(p) = g\sigma$, where g is the conductivity of a single bond and $g_m(p)$ is the EMT result for the p-dependent average conductivity. In particular, Eq. (6) implies that $s_m(p)/\sigma = (2p-1)^{-1}$ in very good agreement with our numerical results for the square lattice ($z = 4$). For both site- and bond-percolation models, the EMT leads to $s_m(p) \sim (p-p_c)^{-1}$, i.e., $\kappa = 1$ in any dimension. As for the conductance, the EMT reproduces the correct qualitative behavior close to p_c, becomes exact for small disorder ($p \sim 1$), but fails to give the exact value of the exponent κ. In this respect, it should be noted that the naive picture, assuming a homogeneous behavior close to $p = 1$, leads to the erroneous result $s(p) \sim 1/p$. The breakdown of such a picture originates from the fact that, even at $p \lesssim 1$, the inhomogeneities of the current pattern are enhanced and lead to a nontrivial behavior of g_m and s_m.

In order to go beyond the EMT approximation and in particular to find estimates for the exponent κ for dimensions (D) other than two, we have used two position-space renormalization-group (PSRG) transformations, introduced previously for the percolation problem[15]: differential PSRG and cell transformation. Using the same notations as Ref. 15, one obtains for the bond-percolation problem[16]

$$\mathcal{L}_{1D}[\langle s \rangle] = -\langle s \rangle, \tag{7a}$$

$$\mathcal{L}_{\parallel}[\langle s \rangle] = \langle s \rangle [I(a)/a - 1], \tag{7b}$$

where $a = p/(1-p)$ and $I(a) = \int_0^1 dx[\ln(1+ax)]/x$. Here L_{1D} and $L_{\parallel}$ are as usual the infinitesimal generators associated with the differential PSRG procedure. Equations (7) lead immediately to the following value of κ/ν: $\kappa/\nu = (d-1)I(a)/a$ taken at $p = p_c$. Using the corresponding values of p_c and ν,[15] one deduces the following: for 2D,

$$p_c = \tfrac{1}{2}, \quad I(a=1) = \pi^2/12:$$

$$\kappa/\nu = 0.822, \quad \kappa = 1.339 \ (\nu = 1.629);$$

for 3D,

$$p_c = 0.16, \quad I(a) = 0.18:$$

$$\kappa/\nu = 1.911, \quad \kappa = 2.332 \ (\nu = 1.22).$$

In the limiting case, $d = 1 + \epsilon$, $\epsilon \ll 1$, one obtains $\kappa/\nu \simeq (1/2\epsilon)\epsilon^{-1/\epsilon}$, in agreement with $\kappa/\nu = 0$ at $d = 1$. On the contrary, for $\epsilon \gg 1$ one obtains $\kappa/\nu \simeq \epsilon$, in disagreement with the large-d result $\kappa/\nu = 4$.

In addition to the differential scheme, we have used a discrete cell transformation, which reduces at $d = 2$ to the iterated "Wheatstone bridge" construction. This method has been used[14] to expand $s(p)$ for $p \lesssim 1$ as well as to calculate it exactly over the whole interval [0.5,1] with Monte Carlo techniques. Close to $p = 1$ we obtain $s(p)/\sigma = 1 + 3(1-p) + O((1-p)^2)$ in contradiction with the EMT result: $s_m(p)/\sigma = 1 + 2(1-p) + O((1-p)^2)$. In this respect, the agreement[17] between this scheme and the EMT results at $p \sim 1$ for $g(p)$ is somewhat accidental. Furthermore, we have performed Monte Carlo calculations, up to eight iterations ($\simeq 10^6$ bonds). The fluctuations in the Monte Carlo data for $g(p)$ and $s(p)$ are very small. From the data near $p_c(=0.5)$, where $s(p)$ and $1/g(p)$ increase sharply, the critical exponents we obtain are $t = 1.32 \pm 0.02$ and $\kappa = 1.70 \pm 0.02$. The value of t is the same as that of Ref. 17. However, the value for κ lies outside the exact bounds for percolation. This overestimation of κ must be compared with that of t and ν. Accordingly, the iterated Wheatstone bridge, when viewed as a hierarchical structure, cannot provide reliable estimates for exponents.

In closing, note that with a correlation function such as the one given by Eq. (1), the problem at hand is unrelated to that of the sample-to-sample variation[18] of the resistance of finite systems with p at the percolation threshold.

Our results have direct implications for experiments on real materials such as metal-insulator mixtures[4] and

possibly Ni-Al_2O_3 cermets at low temperatures,[5] where the measurement of flicker noise may be worthwhile. First, the measurement of $s(p)$ as a function of the metallic filling fraction provides a fine tool for exploration of the geometry of the structure. The magnitude [Eq. (2)] of the relative noise is more strongly affected by the current pattern than the squared conductance. In this respect the critical region ($p \sim p_c$) as well as the region of small disorder ($p \sim 1$) are of great interest. Secondly, the measurement of $s(p)$ will shed some light on the microscopic origin of the flicker noise, be it in the contacts between grains or in intragrain fluctuations.

We would like to thank J. Mantese and M. B. Weissman for useful discussions. Two of us (R.R. and A.-M.S.T.) acknowledge the hospitality of Brookhaven National Laboratory. This work was supported in part by the Natural Sciences and Engineering Research Council of Canada and its program "Attaché de Recherche" and in part by Programme Expérimental de Soutien á l'Emploi Scientifique (Québec). Most calculations were performed on a computer donated by the Imperial Tobacco Company.

Note added.—After this work was submitted for publication, we received a preprint by L. de Arcangelis, S. Redner, and A. Coniglio [Phys. Rev. B **31**, 4725 (1985)] where, for a different purpose, the exponents x_n were also calculated numerically for $n=0$, 0.5, 1, 1.5, and 2. Their results agree with ours. (ζ_{2n}/ν in their notation is $-x_n$ in ours.) We thank S. Redner for letting us know of this work prior to publication.

[1]P. Dutta and P. M. Horn, Rev. Mod. Phys. **53**, 497 (1981), and references therein; F. N. Hooge, T. G. M. Kleinpenning, and L. K. J. Vandamme, Rep. Prog. Phys. **44**, 479 (1981).

[2]R. F. Voss and J. Clarke, Phys. Rev. B **13**, 556 (1976); J. Clarke and R. F. Voss, Phys. Rev. Lett. **33**, 24 (1974); H. G. E. Beck and W. P. Spruit, J. Appl. Phys. **49**, 3384 (1978); A. M. S. Tremblay and M. Nelkin, Phys. Rev. B **24**, 2551 (1981).

[3]R. D. Black, M. B. Weissman, and F. M. Fliegel, Phys. Rev. B **24**, 7454 (1981); J. H. Scofield, D. H. Darling, and W. W. Webb, Phys. Rev. B **24**, 7450 (1981).

[4]D. A. Rudman and J. C. Garland, Bull. Am. Phys. Soc. **29**, 352 (1984).

[5]J. V. Mantese, W. I. Goldburg, D. H. Darling, H. G. Craighead, U. J. Gibson, R. A. Buhrman, and W. W. Webb, Solid State Commun. **37**, 353 (1981).

[6]R. Rammal, C. Tannous, and A.-M. S. Tremblay, Phys. Rev. A **31**, 2662 (1985).

[7]J. Vannimenus and J. P. Nadal, Phys. Rep. **103**, 47 (1984).

[8]M. A. Lemieux, P. Breton, and A.-M. S. Tremblay, J. Phys. (Paris), Lett. (to be published).

[9]H. J. Herrmann, D. C. Hong, and H. E. Stanley, J. Phys. A **17**, L261 (1984).

[10]L. Puech and R. Rammal, J. Phys. C **16**, L1197 (1983); H. J. Herrmann and H. E. Stanley, Phys. Rev. Lett. **53**, 1121 (1984); J. G. Zabolitzky, Phys. Rev. B **30**, 4077 (1984).

[11]Note that in the large-n limit, the inequalities are consistent, for instance, with $x_n = u + vn$, where u and v are constants and v is positive. In the present problem, v vanishes.

[12]A. Coniglio, Phys. Rev. Lett. **46**, 250 (1981).

[13]S. Kirkpatrick, Rev. Mod. Phys. **45**, 574 (1973).

[14]R. Rammal, J. Phys. (Paris), Lett. **46**, L129 (1985).

[15]S. Kirkpatrick, in *Ill-Condensed Matter,* Proceedings of the Les Houches Summer School, Session XXX, edited by R. Balian, R. Maynard, and G. Toulouse (North-Holland, Amsterdam, 1979).

[16]S. Kirkpatrick, Phys. Rev. B **15**, 1533 (1977).

[17]J. Bernasconi, Phys. Rev. B **18**, 2185 (1978).

[18]A. B. Harris, S. Kim, and T. C. Lubensky, Phys. Rev. Lett. **53**, 743 (1984); R. Rammal, M.-A. Lemieux, and A.-M. S. Tremblay, Phys. Rev. Lett. **54**, 1087 (1985).