Statistical Theory of Cubic Langmuir Turbulence

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The cubic direct-interaction approximation is applied to the truncated cubically nonlinear Schrödinger equation. The statistical theory does a satisfactory job in several important respects.

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An important mode of Langmuir turbulence in plasma physics,[1–6] as well as deep-water waves in fluid dynamics,[7] is the cubically nonlinear Schrödinger equation

$$i\,\partial_t E(x,t) + \partial_x^2 E + |E|^2 E - WE = 0, \tag{1}$$

where $E(x,t)$ is the dimensionless, slowly varying (in time) amplitude of the rapidly varying (near the electron plasma frequency) Langmuir-wave electric field, t is time, x is distance, and W is the spatial average of $|E|^2$. This equation is, in fact, a generic nonlinear equation which can arise whenever the envelope of a wave train is perturbed.[8] Thus, it is important to seek statistical theories for this equation which rival in accuracy and completeness the statistical theories of fluid turbulence.[9] Unfortunately, the standard weak-turbulence theory of plasma physics does not work at all in this one-dimensional case, predicting no evolution of any initial wave spectrum.[10] Other statistical approaches to strong Langmuir turbulence have been reviewed recently.[6]

In this paper we apply the cubic version[11–17] of Kraichnan's direct-interaction approximation[9,18–20] to a truncated version of (1). That is, we take the electric field to be periodic on $0 \leqslant x \leqslant 2\pi$ so

$$E(x,t) = \textstyle\sum_k E_k(t)\exp(ikx). \tag{2}$$

Then the spatial Fourier transform of (1) is obtained from Hamilton's equation

$$i\,\partial_t E_k = \partial H/\partial E_k^* \tag{3}$$

with the Hamiltonian

$$H = \sum_k |E_k|^2 + \tfrac{1}{2}\left(\sum_k |E_k|^2\right)^2 - \tfrac{1}{2}\sum_{k+k'=k''+k'''} E_k E_{k'} E_{k''}^* E_{k'''}^*. \tag{4}$$

We truncate the system of Eq. (3) so that only modes from $-k_{\max}$ to $+k_{\max}$ are treated, and all summations in (4) and in the remainder of the paper go from $-k_{\max}$ to $+k_{\max}$. In addition to H and $W = \sum_k |E_k|^2$, a third constant of the motion is the "momentum"

$$P = \textstyle\sum_k k|E_k|^2. \tag{5}$$

The cubic direct-interaction approximation involves the set of two-point correlation functions

$$C_k(t,t') = \langle E_k(t) E_k^*(t') \rangle,$$

where angular brackets denote an ensemble average, and the set of response functions $R_k(t,t')$ which give the ensemble-averaged change in $E_k(t)$ due to infinitesimal change in $E_k(t')$ at an earlier time t', divided by the infinitesimal change; $R_k(t,t) = 1$ and $R_k(t,t') = 0$ if $t < t'$. The statistical theory in this case has the form,[14] for each k,

$$\partial_t C_k(t,t') = -ik^2 C_k(t,t') + i[\langle W\rangle - C_k(t,t)]C_k(t,t') + \int_{t_0}^{t'} S_k(t,t'')R_k^*(t',t'')\,dt'' + \int_{t_0}^{t} Z_k(t,t'')C_k(t'',t')\,dt'' \tag{6}$$

and

$$\partial_t R_k(t,t') = -ik^2 R_k(t,t') + i[\langle W\rangle - C_k(t,t)]R_k(t,t') + \int_{t'}^{t} Z_k(t,t'')R_k(t'',t')\,dt'', \tag{7}$$

where the "nonlinear noise source" is

$$S_k(t,t'') = |C_k|^2 C_k - 3C_k\sum_{k'}|C_{k'}|^2 + 2\sum_{k'k''} C_{k'}C_{k''}C^*_{-k+k'+k''} \tag{8}$$

and the "self-energy" is

$$Z_k(t,t'') = -2R_k|C_k|^2 + R_K^* C_k^2 + 3R_k\sum_{k'}|C_{k'}|^2 + 3C_k\sum_{k'}R_{k'}C_{k'}^* - 3C_k\sum_{k'}R_{k'}^* C_{k'} - 4\sum_{k'k''}R_{k'}C_{k''}C^*_{-k+k'+k''} + 2\sum_{k'k''}R_{k'}^* C_{k''}C_{k+k'-k''}; \tag{9}$$

all time arguments on the right-hand sides of (8) and (9) are the same as those on the left, and all summations include only those subscripts in the range $-k_{\max}$ to $+k_{\max}$. It has been assumed that the initial-value ensemble has $\langle E_k \rangle = \langle E_k E_{k'} \rangle = 0$ for all k, k', and that it is Gaussian. It is shown in Sun, Nicholson, and Rose[21] and Sun[22] where further details of this work can be found, that the statistical theory conserves $\langle W \rangle$, $\langle P \rangle$, and $\langle H \rangle$.

There exists a class of stationary solutions to (6) and (7) in which $C_k(t,t') = C_k(t-t')$ and $R_k(t,t') = R_k(t-t')$ for all k; in general, these require $t_0 \to -\infty$. These "thermal equilibrium" solutions correspond to an ensemble in which the one-time probability distribution function is given by

$$P(\{E_k\}) = K \exp(-\alpha W - \beta H - \gamma P), \qquad (10)$$

where K is a normalization constant; this distribution is normalizable only for finite $k_{\max}$ and for certain ranges of the real temperatures α, β, and γ. A straightforward extension of the calculation of Deker and Haake[23,24] leads to the "fluctuation-dissipation" theorem

$$R_k(t-t') = (i\beta \partial_t + \alpha + \gamma k) C_k(t-t') \qquad (11)$$

for $t > t'$, and a corresponding relation between the nonlinear noise source and the self-energy,

$$Z_k(t-t') = -(i\beta \partial_t + \alpha + \beta k) S_k(t-t') \qquad (12)$$

for $t > t'$. By using (11) and (12) in the statistical theory (6) at $t' = t$, one obtains the consistency relation

$$1 - (\alpha + \gamma k) C_k(0) = \beta [k^2 - \langle W \rangle + C_k(0) - \beta S_k(0)] C_k(0), \qquad (13)$$

which is valid for each k. The significance of this discussion is that, for arbitrary initial values $\{C_k(t_0,t_0)\}$, our numerical solutions of the statistical theory (6) and (7) evolve to one of the stationary solutions represented by (10), and the stationary values $\{C_k(0)\}$ are predicted by (13) plus the conservation laws.

Let us now present the results of our numerical solution of the statistical theory (6) and (7). In Fig. 1 the numerical solutions of (6) and (7) for $C_k(t=0,t')$, $k=0$, 1, 2, are compared to the corresponding averages for an initially Gaussian ensemble of 10 000 numerical solutions of the dynamical equation (3) in the five-mode ($k_{\max}=2$) case, with the initial conditions $C_0(0,0) = C_1(0,0) = C_2(0,0) = 1$. (In all calculations in this paper, $C_{-k} = C_k$ for all k and at all times.) The error bars in Fig. 1(a) represent the standard error for the quantity $|E_0(0) E_0^*(t')|$. Figures 1(a) and 1(b) show that the theory does a remarkably good job of predicting $|C_0(0,t')|$ and $|C_1(0,t')|$ in the case, especially for $0 \le t' \le 1$. The agreement for $|C_2(0,t')|$ in Fig. 1(c) is not as good, but is still qualitatively quite satisfactory for statistical theories of this type. This example, both analytically through (10)–(13) and numerically, corresponds to a stationary state in which $C_k(t,t') = C_k(t-t')$ for t, $t' > 0$, $k=0$, 1, 2, which we call an equipartition state. The appropriate temperatures are $\alpha = 1$, $\beta = 0$, $\gamma = 0$.

In Fig. 2 the equal-time functions $C_0(t,t)$ and $C_1(t,t)$ obtained from the statistical theory (6) and (7) are compared to an ensemble of numerical solutions of the dynamical equation (3), for the three-mode ($k_{\max}=1$) case with initial conditions $C_0(0,0) = 10$, $C_1(0,0) = 1$. Both the ensemble and the statistical theory reach an approximate stationary state after a

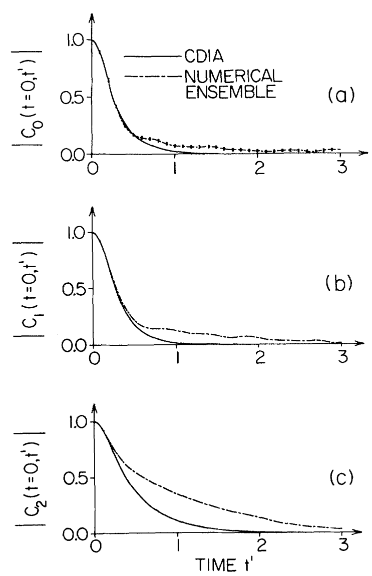

FIG. 1. Numerical solution of the statistical theory (6) and (7) compared to an ensemble of numerical solutions of (3) in the five-mode ($k_{\max}=2$) case, with initial conditions $C_0(0,0) = C_1(0,0) = C_2(0,0) = 1$. (a) Two-time correlation function $|C_0(0,t')|$. (b) $|C_1(0,t')|$. (c) $|C_2(0,t')|$.

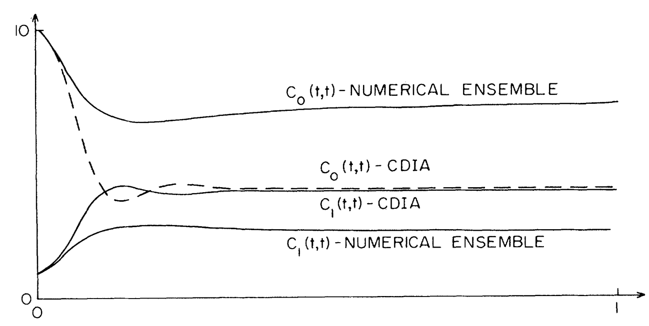

FIG. 2. Single-time correlation functions $C_0(t,t)$ and $C_1(t,t)$ from the numerical solution of the statistical theory (6) and (7) compared to those of an ensemble of numerical solutions of (3) in the three-mode ($k_{\max}=1$) case, with initial conditions $C_0(0,0) = 10$, $C_1(0,0) = 1$.

time of order 0.5. The stationary values are not in very good agreement: the ensemble has

$$\lim_{t\to\infty} C_0(t,t) \gg \lim_{t\to\infty} C_1(t,t)$$

(this result is easily understood[21,22] on the basis of the standard integrable three-mode problem[25–27] well known in plasma physics) while the statistical theory reaches a state which is almost but not quite in equipartition,

$$\lim_{t\to\infty} C_0(t,t) \approx \lim_{t\to\infty} C_1(t,t).$$

The statistical theory does a better job in Fig. 3 under the same conditions as in Fig. 2 but with the initial conditions $C_0(0,0)=0.1941$ and $C_1(0,0)=0.4659$. The statistical theory and the numerical ensemble both appear to evolve to stationary states which are not too different from each other. By considering $\langle W\rangle$ and $\langle H\rangle$ in the initial Gaussian state, and in the final thermal equilibrium state [$\langle P\rangle=0$ and $\gamma=0$ throughout this paper since $C_{-k}(t,t')=C_k(t,t')$ for all k], and by writing (13) for $k=0$ and $k=1$, one obtains high-order algebraic equations relating the initial conditions $C_0(0,0)$ and $C_1(0,0)$ to the final stationary values $C_0\equiv C_0(0)$ and $C_1\equiv C_1(0)$ and the temperatures α and β. In the present case, the initial values chosen correspond to the predicted final values $\alpha=2.874$, $\beta=-1$, $C_0=0.2941$, $C_1=0.4159$, which indeed correspond to Fig. 3 to within numerical accuracy.

It is possible to compare the properties of the stationary solutions of the statistical theory with the corresponding exact consequences of the probability distribution (10). In the three-mode ($k_{max}=1$) case we write the partition function ($\gamma=0$)

$$Z=\int dE_0\, dE_1\, dE_{-1} \exp(-\alpha W-\beta H), \tag{14}$$

where dE_0 represents a two-dimensional integration over real and imaginary parts, $dE_0=dE_{0r}dE_{0i}$, and all integrations go from $-\infty$ to $+\infty$. In terms of the partition function,

$$\langle H\rangle = -\partial \ln Z/\partial\beta. \tag{15}$$

Four of the six integrations in (14) can be done analytically[28]; performing the remaining two numerically we obtain $\langle H\rangle$ vs β at fixed α. For $\alpha=6\pi$, this curve is labeled "EXACT" in Fig. 4. It is compared in that figure to the corresponding curve, labeled "CDIA," obtained as in the discussion of Fig. 3. For completeness, we also display the predictions of two simpler theories. The "HARTREE" approximation is the same as the cubic direct-interaction approximation except that $S_k(0)$ in (13) is set equal to zero for all k. The "WEAK TURBULENCE" approximation is obtained by ignoring all terms higher than quadratic in the Hamiltonian (4); (15) then yields $\langle H\rangle=2/(\alpha+\beta)$ as shown in Fig. 4.

Note that the comparisons in Fig. 4 are only among the probability distribution (10) and the stationary solutions of the statistical theories; in the three-mode case the actual numerical ensemble involves integrable, periodic motion in each realization and there is no reason to expect that an initial Gaussian ensemble will evolve to a thermal equilibrium state characterized by a probability distribution of the form (10).

We have also made an extensive investigation of the case where the Hamiltonian equations (3) are supplemented with linear damping and driving terms. The dynamics of this case have been treated elsewhere[25,26] in detail. We find that the statistical theory also yields

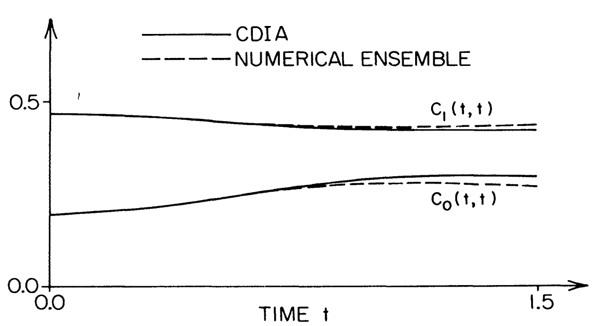

FIG. 3. Same as Fig. 2 with initial conditions $C_0(0,0)=0.1941$, $C_1(0,0)=0.4659$.

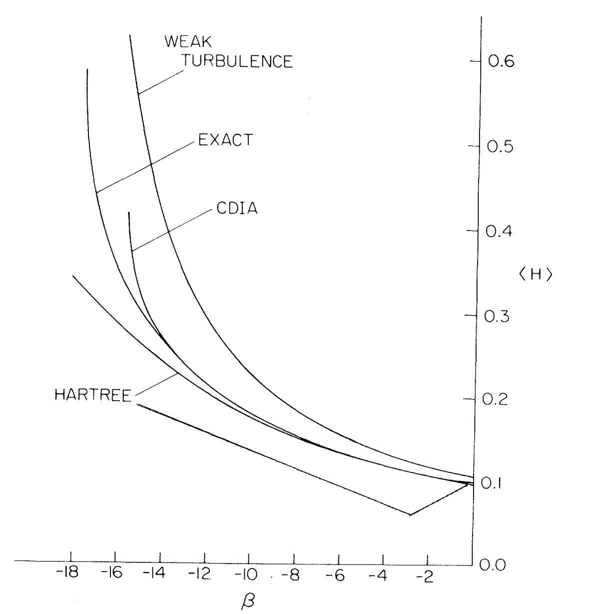

FIG. 4. Comparison of the "exact" numerical evaluation of (14) and (15) with the corresponding predictions of three statistical theories for $\alpha=6\pi$.

reasonable results in this case, evolving to a stationary state for a broad range of driving and damping; we discuss the detailed interpretation of these results elsewhere.[21,22]

We conclude that the cubic direct-interaction approximation yields satisfactory qualitative, and in many cases quantitative, results when applied to the cubically nonlinear Schrödinger equation truncated to a few modes. There is some indication that the results improve with more modes. It remains for future work to determine whether present computational resources are capable of solving the theory when the number of modes is large enough (50 or 100) to reproduce the partial differential Eq. (1) in physically interesting cases.

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