Higher-Order Correlations in Spectra of Complex Systems

Oriol Bohigas
Division de Physique Théorique, Institut de Physique Nucléaire, 91406 Orsay Cedex, France

and

Rizwan U. Haq
Department of Physics and Astronomy, Laurentian University, Sudbury, Ontario P3E 2C6, Canada

and

Akhilesh Pandey
Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627
(Received 21 September 1984)

The presence and significance of three- and four-level correlation effects in nuclear spectra are demonstrated for the first time. New ways to study two-level properties are also presented. Agreement with random-matrix predictions is found.

PACS numbers: 24.60.Ky, 05.40.+j

The two-level correlation function for the spectra of complicated systems[1] determines essentially all of the fluctuation measures which have so far been seriously studied. Moreover, it determines two general properties, namely the Wigner–von Neumann level repulsion (tendency of levels to avoid clustering) and the Dyson-Mehta *long-range order* (large correlations between distant levels). In an earlier analysis,[2] by combining all the high quality nuclear data into a *nuclear data ensemble* (NDE) and by introducing new spectrally averaged measures, we found an extremely close agreement with the *Gaussian orthogonal ensemble* (GOE) predictions for two-level correlations. This goes a considerable distance toward confirming Wigner's suggestion[3] that "the Hamiltonian which governs the behavior of a complicated system is a random symmetric matrix, with no particular properties except for its symmetric nature." Because of that, and because of recent interest in problems of quantum extensions of classically chaotic motion,[4] it is important to ask whether the GOE, which follows naturally from Wigner's proposition for time-reversal–invariant systems, gives proper predictions for fluctuations of higher order, say three- and four-level correlations. These of course are not determined by the two-level function, even though a lower-order function does impose certain complicated constraints which are largely unknown on the higher-order ones.

Consider an infinite stationary spectrum with unit average spacing. The k-level correlation function[5] $R_k(r_1, \ldots, r_k)$ is the probability density of observing a level at each of the k points $r_1, \ldots, r_k$, irrespective of the location of the other levels. The R_k's for $k > 1$ ($R_1 = 1$) depend only on the relative variables $r_i - r_j$, e.g., $R_2(r_1,r_2) = R_2(r_1 - r_2)$. The probability $E_k(r)$ that the number statistic $n(r)$, the number of levels in an interval of length r, takes the value k is given by[6]

$$E_k(r) = \frac{1}{k!}\sum_{l=0}^{\infty}(-1)^l\frac{\hat{R}_{k+l}(r)}{l!}, \qquad (1)$$

$$\hat{R}_k(r) = \int_0^r \cdots \int_0^r R_k(r_1, \ldots, r_k)\,dr_1 \cdots dr_k. \qquad (2)$$

For small r, $\hat{R}_k/k!$ gives the probability that in an interval of length r there are k levels. From (1) we have[1,6]

$$\bar{n} = r, \qquad (3)$$

$$\Sigma^2(r) = \hat{R}_2(r) - r(r-1), \qquad (4)$$

$$\kappa_3(r) = \gamma_1(r)\Sigma^3(r) = \hat{R}_3(r) - 3(r-1)\Sigma^2(r) - r(r-1)(r-2), \qquad (5)$$

$$\kappa_4(r) = \gamma_2(r)\Sigma^4(r) = \hat{R}_4(r) - (4r-6)\kappa_3(r) - 3\Sigma^4(r) - (6r^2 - 18r + 11)\Sigma^2(r) - r(r-1)(r-2)(r-3), \qquad (6)$$

for the average, variance, and third and fourth cumulants of n, respectively; γ_1, γ_2 defined in (5), (6) are the skewness and excess, respectively. Note that in general the kth cumulant depends on all $\hat{R}_\nu(r)$ with $\nu \leq k$ so that Σ^2, κ_3, and κ_4 derive from the 2-, (2+3)-, and (2+3+4)-level functions, respectively.

To understand the distinct role played by level repulsion and long-range order on the k-level fluctuation measures we shall, apart from GOE, also consider the following: (i) *Poisson ensemble*: This is an ensemble characterized by an exponential $[\exp(-x)]$ nearest-neighbor spacing distribution $p(x)$ and no correlations between spacings. One has $R_k = 1$ for all

TABLE I. Small-r behavior of $\hat{R}_2$, $\hat{R}_3$, and $\hat{R}_4$.

	GOE	Poisson	UW	GUE
$\hat{R}_2(r)$	$\pi^2 r^3/18$	r^2	$\pi r^3/6$	$\pi^2 r^4/18$
$\hat{R}_3(r)$	$\pi^4 r^6/1350$	r^3	$\pi^2 r^5/80$	$\pi^6 r^9/48\,600$
$\hat{R}_4(r)$	$\dfrac{\pi^8 r^{10}}{6\,615\,000}$	r^4	$\dfrac{\pi^3 r^7}{1680}$	$\dfrac{\pi^{12} r^{16}}{53\,581\,500\,000}$

k, so that $\hat{R}_k = r^k$, and $\Sigma^2(r) = \kappa_3(r) = \kappa_4(r) = r$. This ensemble exhibits neither level repulsion nor long-range order. (ii) *Uncorrelated Wigner (UW) ensemble*: There are, as in (i), no correlations between spacings but the spacing distribution $p(x)$ is that of GOE, for which the Wigner surmise $[(\pi x/2)\exp(-\pi x^2/4)]$, used here, is an excellent approximation. This ensemble exhibits no long-range order but it does show level repulsion and in fact has the same small-r behavior as that of GOE for $R_2(r)$. (iii) *Gaussian unitary ensemble* (GUE): This is an ensemble of complex Hermitian matrices, valid, unlike GOE, for systems in which time-reversal invariance is completely broken. This ensemble exhibits both features more strongly than GOE. Poisson and UW, unlike GOE and GUE, are not matrix ensembles.

GOE and GUE predictions are known in closed form[1] for $k=2$. For $k>2$ we have obtained the results from the tabulated values of the E_k functions.[7] GOE-sample errors needed for comparison with data of limited size (such as the NDE) have been estimated from Monte Carlo calculations. The errors for other ensembles are in general larger than in GOE for Poisson and UW, and smaller for GUE. For the experimental data analyzed here, an NDE of 1762 levels, see Ref. 2.

We discuss first a new analysis of the R_2 function providing efficient measures of level repulsion and long-range order. For level repulsion it is necessary that $R_2(r)$ be zero at $r=0$ [implying that $p(x)=0$ at the origin] and the slope is then a measure of level repulsion. Thus by considering $R_2(r) = \alpha + \beta r$ for small r, so that $\hat{R}_2(r) = \alpha r^2 + (\beta/3) r^3$, we can estimate α and β from data [see Tables I and II and Fig. 1(a)]. Using $r=0.25$ and 0.5 we obtain $\alpha = -0.02$ (consistent with zero) and $\beta = 1.56$ (consistent with the GOE value $\pi^2/6$ to within $\sim 10\%$). To measure long-range order, consider

$$\Sigma^2(r) = ar + 2b\ln r + c \tag{7}$$

valid for the above ensembles for $r \gtrsim 1$. In (7), a is given by

$$a = 1 - \int_{-\infty}^{+\infty} Y_2(s)\,ds, \tag{8}$$

where $R_2 = 1 - Y_2$; for long-range order we must have $a=0$, *a condition on the total integral of the two-level cluster function* Y_2 which expresses that the spectrum is "incompressible"[8]; b may be nonzero if Y_2 falls off as b/s^2 (as it does for GOE and GUE) and provides a measure of the long-range order (the smaller b, the more rigid the spectrum is, provided that $a=0$); c becomes independent of r for large r. NDE values of a, b, and c obtained by a least-squares fit of $\Sigma^2(r)$ with (7) in the range $1 \le r \le 25$ are -0.007, 0.11, and 0.45, respectively. They agree with GOE predictions (0 ± 0.005, 0.10 ± 0.02, 0.44 ± 0.02) but disagree with Poisson (1,0,0), UW (0.273,0,0.17), and GUE (0,0.05,0.34) values.

We now turn to the *higher-order correlations.* In Figs. 1(b) and 1(c) are displayed γ_1 and γ_2 as functions of r and in Table II are given values of $\hat{R}_3$ and $\hat{R}_4$. We see that only GOE agrees with the data for the whole range of r and for all quantities.

Let us focus first on the small-r behavior ($r \lesssim 1$). UW and GOE give identical results for $\hat{R}_2$ but differ

TABLE II. Values of $\hat{R}_2$, $\hat{R}_3$, and $\hat{R}_4$ for NDE, GOE ($\pm$ one sample error), UW, and GUE.

	$\hat{R}_2(\times 10^2)$				$\hat{R}_3(\times 10^2)$				$\hat{R}_4(\times 10^2)$			
r	NDE	GOE	UW	GUE	NDE	GOE	UW	GUE	NDE	GOE	UW	GUE
0.25	0.7	0.8±0.1	0.8	0.2								
0.50	5.9	6.4±0.4	6.3	3.0	0.1	0.1±0.1	0.3	0.005				
0.75	20.3	20.3±0.7	20.1	13.2	1.2	1.1±0.3	2.3	0.1				
1.0	44.8	44.6±0.9	45.2	34.4	5.8	5.4±0.8	10.2	1.1	0.1	0.1±0.1	1.3	$\sim$0.001
1.5	128.5	127.6±1.3	133.0	113.8	48.2	45.3±2.3	68.2	21.9	3.2	4.2±1.4	22.0	$\sim$0.3
2.0	259.0	258.4±1.5	211.2	241.6	180.7	178.0±4.8	239.4	125.5	44.7	46.8±5.0	144.3	$\sim$12

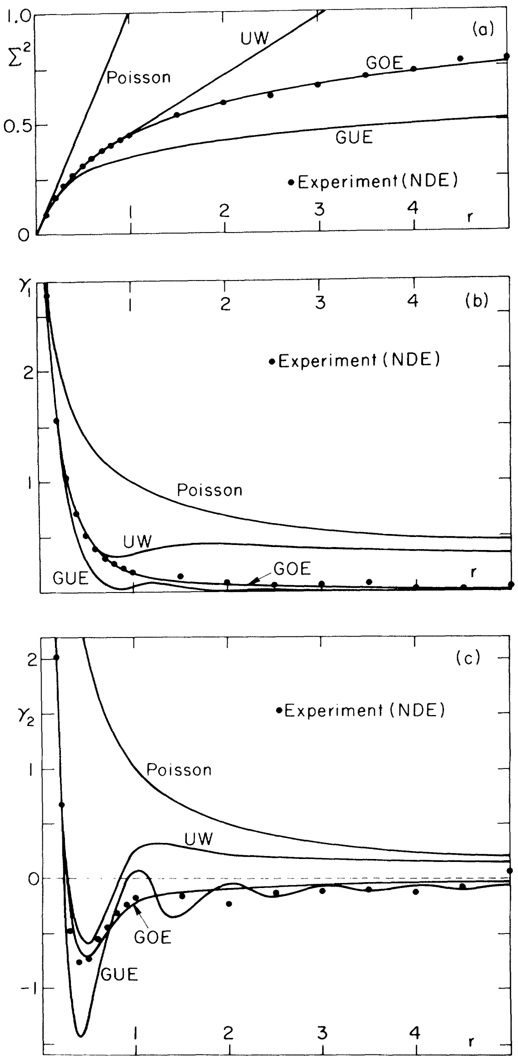

FIG. 1. (a) $\Sigma^2(r)$, (b) $\gamma_1(r)$, and (c) $\gamma_2(r)$ for $r \leqslant 5$. The sample errors corresponding to GOE theory are as follows. For Σ^2, they rise monotonically from 0.001 for $r=0.25$, to 0.009 for $r=1$, and 0.03 for $r=5$. For γ_1, they are of the order of 0.02 for $0.25 \leqslant r \leqslant 1.5$ and then rise to 0.05 for $r=5$. For γ_2, they decrease from 0.08 for $r=0.25$ to 0.05 for $r=1.5$ and then rise again to 0.08 for $r=5$. Although not shown in the figure, the results for $5 \leqslant r \leqslant 25$ have also been calculated, and NDE and GOE values are consistent.

for $\hat{R}_3$ and $\hat{R}_4$. In fact (cf. Table I) as $r \to 0$, $\hat{R}_2$ goes to zero in the same way but $\hat{R}_3$ and $\hat{R}_4$ go to zero more rapidly for GOE than UW. The difference can be attributed to intrinsic higher-order correlation effects. Experimental values (cf. Fig. 1 and Table II) do distinguish between UW and GOE for γ_1, γ_2, $\hat{R}_3$, and $\hat{R}_4$, showing therefore three- and four-level effects of the GOE type in the data.

Consider finally the large-r ($r \gtrsim 1$) results. γ_1 and γ_2 go to zero for all ensembles, indicating that the distribution of $n(r)$ becomes Gaussian. The approach to zero, however, is faster for GOE and data (and even more for GUE) than for Poisson and UW. In fact κ_3 and κ_4 become small for GOE (and GUE), not ruling out the possibility that they go to zero. One can prove that for Poisson and UW the leading term in the expansion of κ_3 and κ_4 is linear in r, whereas for GOE and GUE it will be at most logarithmic. This is a consequence of the relations[8] satisfied by the total integrals of the k-level cluster functions and may be regarded as an aspect of long-range order in the three- and four-level correlations.

We have thus presented a new analysis of the two-level correlation properties and introduced precise measures of level repulsion and long-range order. In particular, the total integral of the two-level cluster function, essential for determining long-range order, is deduced from the data. We have also introduced methods for analyzing higher-order correlations. We have shown for the first time that the three- and four-level effects predicted by the GOE are present in the nuclear resonance data.

Significant progress in understanding Wigner's prescription for complicated systems has been made recently in the form of the following conjecture[4] concerning the relationship between classical and quantum chaos: Generically an integrable quantum system should exhibit Poisson-type energy-level fluctuations; on the other hand, quantization of nonintegrable time-reversal–invariant systems should yield fluctuations intermediate between Poisson and GOE, the latter applying for systems for which almost all classical trajectories are chaotic. The conjecture is well supported by numerical calculations for two-dimensional model Hamiltonians.[4] The best experimental evidence for level fluctuations comes from nuclear spectra,[2] which we have used, but there is also evidence from atomic[9] and molecular[10] spectra and from the low-temperature properties of small metallic particles.[1] All of these tests have centered on the two-level correlation properties. Present results confirm the validity of the GOE even for higher-order correlations and go a considerable distance toward showing the "universality" of the level fluctuations, which is implied by Wigner's prescription and the above conjecture.

We are indebted to J. B. French for many valuable discussions and for critical comments on the manuscript. This work has been partly carried out while two of the authors (O.B. and R.U.H.) were visiting the Department of Physics and Astronomy, University of Rochester; they acknowledge the hospitality extended to them. This work has been supported in part by the U. S. Department of Energy and the Natural Sciences and Engineering Research Council of Canada. Division de Physique Théorique is a labora-

toire associé au Centre National de la Recherche Scientifique.

[1]T. A. Brody, J. Flores, J. B. French, P. A. Mello, A. Pandey, and S. S. M. Wong, Rev. Mod. Phys. **53**, 385 (1981), and references therein.

[2]R. U. Haq, A. Pandey, and O. Bohigas, Phys. Rev. Lett. **48**, 1086 (1982); O. Bohigas, R. U. Haq, and A. Pandey, in *Nuclear Data for Science and Technology,* edited by K. H. Böckhoff (Reidel, Dordrecht, 1983), p. 809; the data (NDE) used in the present analysis are the same as in the second paper which contains a few more sequences than the first one.

[3]E. P. Wigner, in *Symmetries and Reflections,* edited by E. P. Wigner (Ox Bow, Woodbridge, Connecticut, 1979), p. 203.

[4]I. C. Percival, J. Phys. B **6**, L229 (1973); M. V. Berry and M. Tabor, Proc. Roy. Soc. London, Ser. A **356**, 375 (1977); O. Bohigas, M. J. Giannoni, and C. Schmit, Phys. Rev. Lett. **52**, 1 (1984); T. H. Seligman, J. J. Verbaarschot, and M. R. Zirnbauer, Phys. Rev. Lett. **53**, 215 (1984); M. V. Berry, in *Chaotic Behavior of Deterministic Systems,* edited by G. Iooss *et al.* (North-Holland, Amsterdam, 1983); O. Bohigas and M. J. Giannoni, in *Mathematical and Computational Methods in Nuclear Physics,* edited by J. S. Dehesa *et al.*, Lecture Notes in Physics Vol. 209 (Springer-Verlag, New York, 1984), and references therein.

[5]F. J. Dyson, Commun. Math. Phys. **19**, 235 (1970); M. L. Mehta, Commun. Math. Phys. **20**, 245 (1971). These papers contain expressions for R_k for GOE and GUE.

[6]A. Pandey, Ph.D. thesis, University of Rochester, 1978 (unpublished).

[7]M. L. Mehta and J. des Cloizeaux, Indian J. Pure Appl. Math. **3**, 329 (1972).

[8]M. L. Mehta and F. J. Dyson, J. Math. Phys. **4**, 713 (1963).

[9]N. Rosenzweig and C. E. Porter, Phys. Rev. **120**, 1698 (1960); H. S. Camarda and P. D. Georgopulos, Phys. Rev. Lett. **50**, 492 (1983).

[10]E. Haller, H. Köppel, and L. S. Cederbaum, Chem. Phys. Lett. **101**, 215 (1983); E. Abramson, R. W. Field, D. Imre, K. K. Innes, and J. L. Kinsey, J. Chem. Phys. **80**, 2298 (1984); S. Mukamel, J. Sue, and A. Pandey, Chem. Phys. Lett. **105**, 134 (1984).