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Three-Cocycle in Mathematics and Physics

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It is shown that the three-cocycle arises when a representation of a transformation group is nonassociative, so that the Jacobi identity fails. A physical setting is given: When the translation group in the presence of a magnetic monopole is represented by gauge-invariant operators, a (trivial) three-cocycle occurs. Insisting that finite translations be associative leads to Dirac's monopole quantization condition. Attention is called to the possible relevance of three-cocycles in the quark model's U(6) ⊗ U(6) algebra.

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A unified mathematical point of view towards various aspects of an anomalous[1] (i.e., apparently inconsistent) gauge theory has now been established[2]: Both the anomalous divergence of the gauge current[1,3] and the anomalous commutators of generators of local gauge transformations[4] are, respectively, the first and second (infinitesimal) cocycles. In this Letter I explain the function of a three-cocycle in group-representation theory, and exhibit physical systems that make use of it.

Consider representing the action of a transformation group G on quantities Ψ depending on a variable q, which transforms under G to q^g, where g is a member of G. We suppose that the action of the representation $U(g)$ involves an operator $\mathscr{A}_1$, which enters as

$$U(g)\Psi(q)=\mathscr{A}_1(q;g)\Psi(q^g). \tag{1}$$

The operators $\mathscr{A}_1$ do not necessarily follow the group composition law, $g_1g_2=g_{12}$; rather they can satisfy

$$\mathscr{A}_1(q;g_1)\mathscr{A}_1(q^{g_1};g_2) =\mathscr{A}_2(q;g_1,g_2)\mathscr{A}_1(q;g_{12}). \tag{2}$$

Here $\mathscr{A}_2$ is another operator, which commutes and associates with $\mathscr{A}_1$. Finally, we assume that the composition law for $\mathscr{A}_1$ is not associative: Different ways of associating a triple product differ by a phase, as

$$[\mathscr{A}_1(q;g_1)\mathscr{A}_1(q^{g_1};g_2)]A_1(q^{g_{12}};g_3)=\exp[i\alpha_3(q;g_1,g_2,g_3)]\mathscr{A}_1(q;g_1)[\mathscr{A}_1(q^{g_1};g_2)\mathscr{A}_1(q^{g_{12}};g_3)]. \tag{3}$$

In order that (3) be consistent with nonvanishing α_3, that phase must satisfy a certain condition. To find the condition, we multiply (3) on the right by $\mathscr{A}_1(q^{g_{123}};g_4)$ and repeatedly use (2) and (3) to bring the association of the four factors in both elements of the equality into the same form. We then find that α_3 must satisfy the three-cocycle condition:

$$\alpha_3(q^{g_1};g_2,g_3,g_4)-\alpha_3(q;g_{12},g_3,g_4)+\alpha_3(q;g_1,g_{23},g_4)-\alpha_3(q;g_1,g_2,g_{34})+\alpha_3(q;g_1,g_2,g_3)\equiv 0(\mathrm{mod}2\pi). \tag{4}$$

A three-cocycle is trivial if it can be written as

$$\alpha_3(q;g_1,g_2,g_3)=\omega(q^{g_1};g_2,g_3)-\omega(q;g_{12},g_3)+\omega(q;g_1,g_{23})-\omega(q;g_1,g_2), \tag{5}$$

where ω is an arbitrary quantity. When (5) holds, α_3 may be removed by redefinition of $\mathscr{A}_2$. It is clear that if $\mathscr{A}_2$ is a number rather than an operator, α_3 is trivial. Moreover, $\mathscr{A}_1$ cannot be a well-defined linear operator—these

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always associate.

These formulas are equivalent to

$$U(g_1)\,U(g_2)\Psi(q)=\mathscr{A}_2(q;g_1,g_2)\,U(g_{12})\Psi(q), \tag{6}$$

and

$$[U(g_1)\,U(g_2)]\,U(g_3)\Psi(q)=\exp[i\alpha_3(q;g_1,g_2,g_3)]\,U(g_1)[U(g_2)\,U(g_3)]\Psi(q), \tag{7}$$

which together with (1) define a "nonassociative representation."

Next, we examine the implication of all this for infinitesimal, algebraic relations. The group element is represented by $g=\exp(\theta^a T^a)$, where θ^a is the infinitesimal parameter, and T^a is an element of the Lie algebra satisfying the commutation relation

$$[T^a,T^b]=f_{abc}T^c. \tag{8}$$

(Summation over repeated indices is implied.) Although the representation of the group is nonassociative, we assume that it is power associative, i.e., n-fold products of a given quantity are uniquely defined. This allows the expression of representatives of finite group elements $U(g)$ in terms of infinitesimal generators $G_\theta\equiv\theta^a G_a$:

$$U(g)=\exp G_\theta=1+G_\theta+\dots. \tag{9}$$

The exponential is well defined by its power series. Also, we define the lowest-order quantities

$$\mathscr{A}_2(q;g_1,g_2)=I+\frac{1}{2}\mathscr{A}_{\theta_1\theta_2}(q)+\dots, \tag{10}$$

$$\alpha_3(q;g_1,g_2,g_3)=\frac{1}{3!}\alpha_{\theta_1\theta_2\theta_3}(q)+\dots. \tag{11}$$

The commutator of two generators follows from (6):

$$[G_{\theta_1},G_{\theta_2}]=G_{\theta_1\times\theta_2}+\mathscr{A}_{\theta_1\theta_2},$$
$$(\theta_1\times\theta_2)^a\equiv f_{abc}\theta_1^b\theta_2^c. \tag{12}$$

Multiplication of generators is not associative, since (7) implies that

$$(G_{\theta_1}G_{\theta_2})\,G_{\theta_3}=G_{\theta_1}(G_{\theta_2}G_{\theta_3})+\frac{i}{3!}\alpha_{\theta_1\theta_2\theta_3}, \tag{13}$$

which in turn has the consequence that the Jacobi identity fails:

$$[[G_{\theta_1},G_{\theta_2}],G_{\theta_3}]+[[G_{\theta_2},G_{\theta_3}],G_{\theta_1}]+[[G_{\theta_3},G_{\theta_1}],G_{\theta_2}]=\frac{i}{3!}\alpha_{[\theta_1\theta_2\theta_3]}. \tag{14}$$

(Here, $[\theta_1\theta_2\theta_3]$ means antisymmetrization in all three quantities.)

In summary, I have shown that a three-cocycle arises in nonassociative group representations as in (7). For power-associative representations, one may define infinitesimal generators which, however, do not satisfy the group's Lie algebra (8); rather, there is an operator extension, as in (12). Finally, the hallmark of all this is that the Jacobi identity is not valid, as in (14).

I now give a physical realization of the above. Although physical cocycles arose first in topologically interesting quantum field theories,[2,5] particularly gauge theories, the example that I offer is drawn from ordinary quantum mechanics involving a magnetic monopole. (I shall mention a possible field-theoretic application later.)

In quantum mechanics one deals with momentum and position operators p^i,r^i which satisfy Heisenberg's commutation relations

$$[r^i,r^j]=0,\quad [r^i,p^j]=i\hbar\delta^{ij},\quad [p^i,p^j]=0. \tag{15}$$

With these one can build a trivial representation of the translation group, representing translations by the exponential of $\mathbf{p}$:

$$\exp[(i/\hbar)\mathbf{a}\cdot\mathbf{p}]\Psi(\mathbf{r})=\Psi(\mathbf{r}+\mathbf{a}). \tag{16}$$

However, when the particle described by these variables carries charge e and moves in the field $\mathbf{B}$ of a magnetic monopole with strength g, located at r_0,

$$\nabla\cdot\mathbf{B}=4\pi g\delta(\mathbf{r}-\mathbf{r}_0), \tag{17}$$

then $\mathbf{p}$ is not gauge invariant; rather, it is related to the gauge-invariant velocity operator $\mathbf{v}$ by a vector potential $\mathbf{A}(\mathbf{r})$ that cannot be globally defined:

$$\mathbf{p}=\mathbf{v}+e\mathbf{A}(r),\quad \nabla\times\mathbf{A}=\mathbf{B}. \tag{18}$$

(The mass of the particle and the velocity of light are set equal to unity.)

Since $\mathbf{v}$ satisfies the same commutation relation with $\mathbf{r}$ as does $\mathbf{p}$,

$$[r^i,v^j]=i\hbar\delta^{ij}, \tag{19}$$

a gauge-invariant representation of translations may be built with the operator

$$U(\mathbf{a})=\exp[(i/\hbar)\mathbf{a}\cdot\mathbf{v}]. \tag{20}$$

However, the velocity components do not commute:

$$[v^i,v^j]=ie\hbar\epsilon^{ijk}B^k, \tag{21}$$

and the Jacobi identify fails[6]:

$$[[v^1,v^2],v^3]+[[v^2,v^3],v^1]+[[v^3,v^1],v^2]=-e\hbar^2\nabla\cdot\mathbf{B}=-4\pi eg\hbar^2\delta(\mathbf{r}-\mathbf{r}_0). \tag{22}$$

Both mathematicians and physicists mostly ignore this violation of the Jacobi identity. The former work on a manifold with one point—the location of the monopole—excluded; the latter observe that wave functions of interest vanish at the monopole. But we retain the full form (22) and recognize that an infinitesimal three-cocycle is encountered. Equation (12) holds with

$$\mathscr{A}_{\theta_1\theta_2}=(-ie/\hbar)\mathbf{a}_1\times\mathbf{a}_2\cdot\mathbf{B}, \tag{23}$$

and for (14) we find

$$\alpha_{[\theta_1\theta_2\theta_3]}=(4\pi eg/\hbar)\mathbf{a}_1\times\mathbf{a}_2\cdot\mathbf{a}_3\delta(\mathbf{r}-\mathbf{r}_0). \tag{24}$$

It remains to understand why the representatives of finite translations (20) do not associate. Before proceeding, let me discuss the numerical coefficient in (22) and (24). According to Dirac, a consistent quantum dynamics for the monopole requires that *eg* be quantized in integer units of $\hbar/2$. Hence, the coefficient of the infinitesimal three-cocycle is in fact $2\pi n$. For the moment, let us ignore this, and remain with an arbitrary value for *eg*.

To recognize the nonassociativity, let us determine $\mathscr{A}_1(\mathbf{r};\mathbf{a})$. We may write (1) as

$$U(\mathbf{a})\Psi(\mathbf{r})=\exp[(i/\hbar)\mathbf{a}\cdot\mathbf{v}]\Psi(\mathbf{r})=\exp[(i/\hbar)\mathbf{a}\cdot\mathbf{v}]\exp[(-i/\hbar)\mathbf{a}\cdot\mathbf{p}]\Psi(\mathbf{r}+\mathbf{a}). \tag{25}$$

The product of the two operators is easily evaluated; one finds[7]

$$\mathscr{A}_1(\mathbf{r};\mathbf{a})=\exp\left[(-ie/\hbar)\int_{\mathbf{r}}^{\mathbf{r}+\mathbf{a}}d\mathbf{s}\cdot\mathbf{A}(\mathbf{s})\right], \tag{26}$$

where the line integral is along the *straight* line joining $\mathbf{r}$ and $\mathbf{r}+\mathbf{a}$. Furthermore, from (2) we see that

$$\mathscr{A}_2(\mathbf{r};\mathbf{a}_1,\mathbf{a}_2)=\exp[-(ie/\hbar)\Phi], \tag{27}$$

where Φ is the outward [direction $\mathbf{a}_1\times\mathbf{a}_2$) flux through the triangle with vertices $(\mathbf{r},\mathbf{r}+\mathbf{a}_1,\mathbf{r}+\mathbf{a}_1+\mathbf{a}_2)$.

Consider now three translations in noncoplanar directions $\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3$; see Fig. 1. Forming the products in (3), we find for the left-hand side

$$\begin{aligned}[\mathscr{A}_1(\mathbf{r};\mathbf{a}_1)\mathscr{A}_1(\mathbf{r}+\mathbf{a}_1;\mathbf{a}_2)]\mathscr{A}_1(\mathbf{r}+\mathbf{a}_1+\mathbf{a}_2;\mathbf{a}_3)&=\exp[(-ie/\hbar)\Phi(ABC)]\mathscr{A}_1(\mathbf{r};\mathbf{a}_1+\mathbf{a}_2)\mathscr{A}_1(\mathbf{r}+\mathbf{a}_1+\mathbf{a}_2;\mathbf{a}_3)\\&=\exp\{(-ie/\hbar)[\Phi(ABC)+\Phi(ACD)]\}\mathscr{A}_1(\mathbf{r};\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3)\end{aligned} \tag{28a}$$

while the right-hand side becomes

$$\begin{aligned}&\exp[i\alpha_3(\mathbf{r};\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3)]\mathscr{A}_1(\mathbf{r};\mathbf{a}_1)[\mathscr{A}_1(\mathbf{r}+\mathbf{a}_1;\mathbf{a}_2)\mathscr{A}_1(\mathbf{r}+\mathbf{a}_1+\mathbf{a}_2;\mathbf{a}_3)]\\&\quad=\exp[i\alpha_3(\mathbf{r};\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3)]\mathscr{A}_1(\mathbf{r};\mathbf{a}_1)\{\exp[(ie/\hbar)\Phi(BCD)]\}\mathscr{A}_1(\mathbf{r}+\mathbf{a}_1;\mathbf{a}_2+\mathbf{a}_3)\\&\quad=\exp[i\alpha_3(\mathbf{r};\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3)]\exp\{(ie/\hbar)[\Phi(BCD)+\Phi(ABD)]\}\mathscr{A}_1(\mathbf{r};\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3).\end{aligned} \tag{28b}$$

Each flux is pointing outward and passes through the triangle specified by the three letters; see Fig. 1. Comparison of the two equations (28) shows that the three-cocycle is $-e/\hbar$ times the total flux emerging from the tetrahedron formed from the three vectors $\mathbf{a}_i$, with one vertex at $\mathbf{r}$. Hence, it is $-4\pi eg/\hbar$ when the monopole is enclosed and zero otherwise. Shrinking the three vectors to produce the infinitesimal cocycle gives rise to the delta function in (22) and (24).

The three-cocycle is trivial in that it equals, as in (5), a sum of terms, each of which is the flux through the appropriate triangle. Nevertheless, if we wish to represent translations by gauge-invariant operators, we must remain with the trivial three-cocycle. Of course, removing it returns the representation to a trivial one in terms of $\mathbf{p}$, as in (16).

Finally, we observe that Dirac's quantization restores associativity of *finite* translations since α_3 becomes $-2\pi n$ or zero, and has no effect in the exponential of (3) and (7). Notwithstanding the associativity of *finite* translations, the infinitesimal cocycle remains as an obstruction to the Jacobi identity because the *infinitesimal* generators do not associate. The argument may be reversed: By demanding that ultimately translations *must be* associative, we *derive* Dirac's quantization of *eg*. This of course, also insures that a globally defined vector bundle exists.

A violation of the Jacobi identity is known in quantum field theory. When the Schwinger term in the commutator between time and space components of a current is a *c*-number, the Jacobi identity for triple commutators of spatial current components must fail.[8] Since deep-inelastic scattering data indicate that the Schwinger term is indeed a *c*-number,[9] consistent with quark-model calculations,[10] the Jacobi identity for spa-

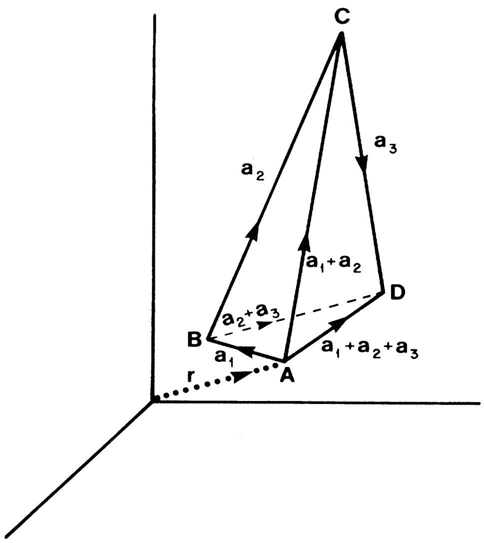

FIG. 1. Tetrahedron at point **r** defined by three translations $\mathbf{a}_i$. The three-cocycle is proportional to the flux out of the tetrahedron.

tial current components should fail in the quark model, and this has been verified in perturbative calculations.[11] The quark-model algebra of time and space components of vector and axial vector currents closes on local $U(6) \otimes U(6)$,[12] and the above remarks indicate that a three-cocycle occurs. However, a well-defined mathematical formulation is problematical, since the Schwinger term very likely is quadratically divergent.[13]

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