Development of Viscous Fingering Patterns

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The development of viscous fingering patterns has been observed from very early times to deep in the nonlinear regime for the flow of immiscible liquids in a Hele-Shaw cell. If the dimensionless viscosity contrast, A, is large, the number of fingers decreases rapidly with time after the initial pattern is established. If $A \simeq 0$, all fingers grow through the entire time range of this experiment. For all values of A there is a power-law relation between the length of the longest finger and the time.

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The formation of viscous fingers in a Hele-Shaw cell represents a particularly simple nonlinear pattern formation problem, and for this reason several recently developed theoretical approaches to the modeling of pattern formation have been applied to this problem.[1-3] This paper presents the results of an experiment in which, by exploitation of the properties of critical binary-liquid mixtures, the crucial parameter, the viscosity contrast, is varied more precisely and over a greater range than ever before and the temporal development of the nonlinear pattern is determined more accurately than in most earlier work. From these experimental improvements two important results have emerged: (1) There is a dramatic difference in interactions between fingers depending on the magnitude of the viscosity contrast, and (2) a viscosity- and surface-tension-independent power law relates the dimensionless length of the longest finger to the dimensionless time.

Hale-Shaw flow is flow between parallel plates whose separation distance, b, is small compared to any other length in the problem. Hill[4] and Saffman and Taylor[5] have demonstrated that the interface between two liquids in a Hele-Shaw cell becomes unstable when the interface between fluids 1 and 2 advances toward fluid 1 at a speed V such that

$$(\rho_1-\rho_2)g+[12(\mu_1-\mu_2)/b^2]V \geqslant 0$$

where μ_i and ρ_i are the shear viscosity and density of the ith liquid. Thus when gravity tends to stabilize the flow ($\rho_1 < \rho_2$), the instability arises only if $\mu_1 > \mu_2$, in which case the critical velocity, V_c, is reduced as the viscosity contrast increases. Chuoke, van Meurs, and van der Poel[6] included interfacial tension, σ, in a linear stability analysis and concluded that the instability was present for all wavelengths greater than

$$\lambda_c = 2\pi[\sigma b^2/12(\mu_1-\mu_2)(V-V_c)]^{1/2}.$$

This analysis also indicated a fastest growing wavelength $=\sqrt{3}\lambda_c$ and experiments tend to show this wavelength during the early stages of the growth of the fingering pattern.[7] Some experiments, including the original work of Saffman and Taylor,[5] suggest that the system reaches a long-time steady-state pattern with only one finger of the less viscous liquid penetrating the other. Theoretical arguments have been raised questioning the stability of any such single-finger solution,[8] but at any rate the early experiments clearly show a finger-amalgamation and -competition mechanism with the late-time pattern exhibiting a significantly larger average wavelength than does the early-time pattern.

Recently Tryggvason and Aref[1] (TA) have performed vortex-sheet calculations to follow the dynamics of this instability far into the nonlinear regime. They cast the problem in terms of two control parameters, a dimensionless viscosity contrast $A=(\mu_2-\mu_1)/(\mu_2+\mu_1)$ and a dimensionless surface tension $B=\sigma b^2/6U^*W^2(\mu_2+\mu_1)$ where W is the width of the Hele-Shaw cell and U^* is a characteristic velocity,

$$U^*=|[(\mu_1-\mu_2)V+\tfrac{1}{12}(\rho_1-\rho_2)gb^2]/(\mu_1+\mu_2)|.$$

A dimensionless time can be defined as $t'=U^*t/WB^{1/2}$ ($B^{1/2}$ is included to scale surface tension out of the Hele-Shaw equations.[1]) Similarly, the length of the mixing zone (θ, the distance along the direction of flow between the tips of the longest fingers in each direction) can be made dimensionless as $\theta'=\theta/WB^{1/2}$. The TA results find a reduction in the number of fingers with time only when A is large. For large A they find some tendency for fingers to amalgamate but, more importantly, long-range interactions along the interface act in such a way that, once one finger gets ahead of the others, the other nearby fingers grow still less rapidly. Neither of these effects occurs for small A in the TA calculations. Most available data[7,9] correspond to $A \sim 1$ (values of A near $\frac{1}{2}$ have usually been achieved by using miscible liquids in which case there is a temporally expanding diffusion zone instead of a sharp interface).

This Letter reports measurements of viscous fingering for a system of immiscible liquids with $A \sim 0$. At the same time measurements were also made for a system with $A \sim 1$ to allow direct comparison. The $A \sim 0$ system used the binary-liquid mixture isobutyric acid plus water at critical composition. The impor-

tant properties of this system are well-known[10] power laws in the reduced temperature $\epsilon = (T_c - T)/T_c$ where T is the temperature and T_c is the system's critical temperature (26.12 °C). Using the published values[10] of $\rho_2 - \rho_1$ and assuming quite reasonably that kinematic viscosity is the same in both phases, one finds that $A = 0.053\epsilon^{\beta}$ and $B = 0.024\epsilon^{2\nu-\beta}$ where $\beta = 0.31$ and $\nu = 0.61$. The temperature of the system was controlled to ± 1 mK and measurements were made at several temperatures between 21.09 °C ($A = 0.015$) and 26.07 °C ($A = 0.004$). The Hele-Shaw cell containing the binary mixture was a square of side length 45 mm with a 1-mm gap. A measurement consisted of preparing the system at equilibrium with a flat, gravity-stabilized meniscus, then inverting the cell and photographing the interface as the nonlinear pattern developed. A digital pattern of the interface was then formed for each picture by running the cursor of a computer-interfaced digitizer over the photograph. A selection from a time series of such patterns is shown in Fig. 1 for a measurement with $A = 0.015$ and $B = 6.6 \times 10^{-4}$. The patterns in Fig. 1 show five fingers; by variation of the temperature, measurements were made for patterns with as many as fifteen fingers. The variation of finger width with temperature followed the $\epsilon^{1/2}$ dependence expected from the arguments above. Figure 2 shows patterns for several different temperatures. Most features of the pattern formation in this liquid mixture even survive the beginning of breakdown in the Hele-Shaw assumptions: As $T \rightarrow T_c$ the characteristic wavelength becomes progressively smaller and eventually becomes comparable to b, at which point a gravitationally unstable Saffman-Taylor instability should begin to change into a Rayleigh-Taylor case. For the $A = 0.0063$ and $A = 0.0039$ cases in Fig. 2 the wavelengths are respectively $3.5b$ and $3.0b$, and the beginning of Hele-Shaw breakdown manifests itself very gently with the formation of two contact lines, one running along each face of the cell. Nevertheless these fingering patterns show a flow which exhibits all the characteristics discussed above for the simple Hele-Shaw case, and also follow the power law to be discussed below.

The $A \sim 1$ system advanced water against paraffin oil in a Hele-Shaw cell of length 1 m, width 36 cm, and gap 0.3 cm. The interface advance was forced by application of a pressure gradient; data were acquired photographically and processed as described above. Figure 3 shows a selection of frames from a time series of patterns for this system with $A = 0.93$ and $B = 8.3 \times 10^{-4}$. It should be noted that, while the $A \sim 1$ system is gravitationally stable and driven unstable by application of a pressure gradient, the $A \sim 0$ system is gravitationally unstable. Since it is experimentally quite difficult to apply an external pressure to the binary-liquid mixture while maintaining the requisite temperature control, the comparison of $A \sim 0$ with $A \sim 1$ depends on there being no essential difference between the two driving mechanisms. Indeed, pressure gradient and gravitational field play equivalent roles in the Hele-Shaw equations and in the charac-

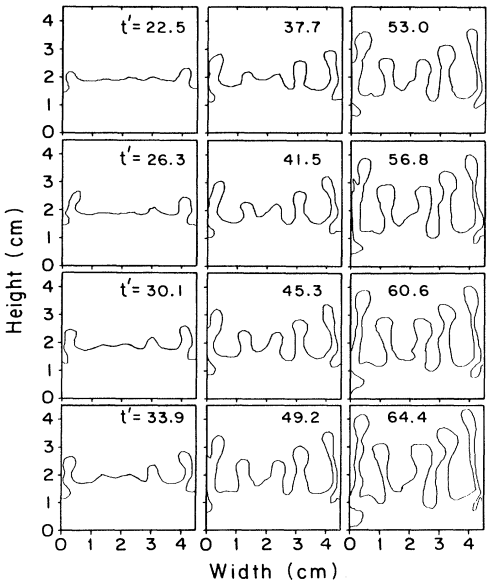

FIG. 1. Time series of fingering patterns for isobutyric acid plus water. $A = 0.015$, $B = 6.6 \times 10^{-4}$. Dimensionless time, t', is indicated for each frame.

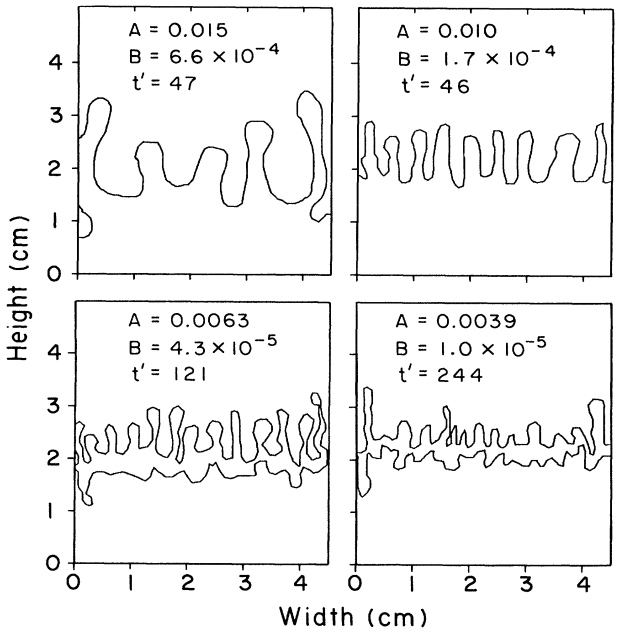

FIG. 2. Typical patterns formed in the low-A system.

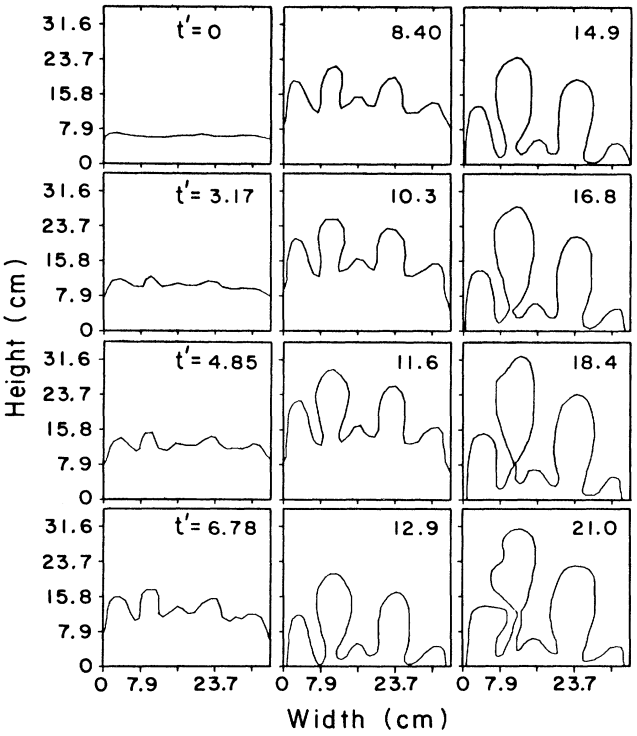

FIG. 3. Time series of fingering patterns for paraffin oil plus water. $A = 0.93$, $B = 8.3 \times 10^{-4}$. The five frames with $t' \geq 12.9$ have their length scale displaced 10 cm downstream from that of the first seven frames.

terestic velocity, U^*, defined above.

The most dramatic differences between high- and low-A systems can be seen by comparing Figs. 1 and 3. The low-A system has never been observed to change the number of fingers during the development of the fingering pattern from the linear regime. Additionally, rapid growth of one finger in the low-A system does not appear to affect its neighbors so that, despite some apparent randomness in the length and width of fingers, there is no systematic impoverishment of fingers in the neighborhood of a rapidly developing finger. Figure 3 shows a high-A case where two fingers grow and the others experience only a very stunted growth. Other high-A measurements (from this experiment but not shown in the figure) have occasionally shown a finger-amalgamation mechanism and always show a stultification of growth of uncompetitive fingers at dimensionless times very small compared to the largest times at which the low-A system has been observed. Since this experiment is designed with wide cells to allow the early-time formation of many fingers and careful observation of these patterns as they develop out of the linear regime, it is not able to answer whether eventually there are steady-state patterns with one or many fingers.

The dimensionless length of the mixing region, θ',

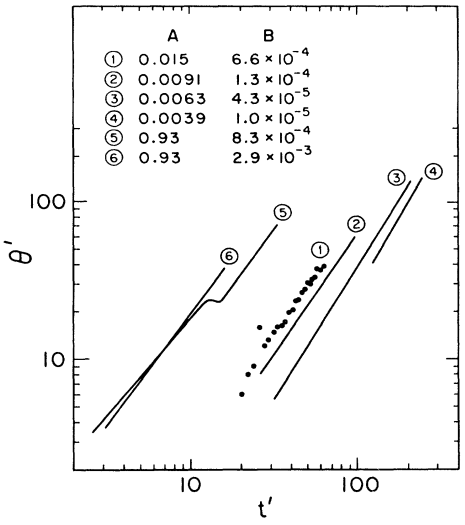

FIG. 4. Dimensionless length of the mixing zone vs dimensionless time as discussed in the text.

shows a remarkably regular dependence on dimensionless time for all mearurements. If one ignores a somewhat noisy early period during which the linear pattern establishes itself, we find $\theta' \sim (t')^{1.6}$ from the emergence of the linear pattern through to the end of the measurement. This can be seen in Fig. 4 where straight lines have been drawn through the data of $\log\theta'$ vs $\log t'$ for several measurements and the individual data are shown for one additional measurement. There is some scatter in the slopes of these lines about their average value of ~ 1.6, but this does not seem large when one considers that the uncertainty in any one slope is relatively large since the θ' measurements span less than two decades. One of the measurements on paraffin oil plus water ($A = 0.93$, $B = 8.3 \times 10^{-4}$) shows an abrupt shift in the θ' vs t' curve at a rather late time. This shift results from the water abruptly displacing paraffin oil along the Teflon spacers at the side of the cell after a long period of stasis at the side wall. No such shift ever appeared in any measurement with the binary-liquid system; the two phases of the binary-liquid systems have such similar composition that their wetting properties at the walls are very similar and no sticking was ever observed. Figure 1 does show a slightly earlier growth of fingers near the side walls for the binary-liquid mixture. This tendency was seen for all the $A \sim 0$ measurements and has been reported in previous $A \sim 1$ measurements.[9] It is difficult to assess whether or not this wall effect should always be present; it is smaller in the present $A \sim 1$ measurements than in Ref. 9 and smaller still in the $A \sim 0$ patterns of Fig. 2. A related question which has recently been addressed[11] involves the appropriate pressure drop across a curved interface, but even Ref.

11 ignores variations in local interfacial curvature due to variations of the contact angle with velocity.

The power law $\theta' \sim (t')^{1.6}$ corresponds to an acceleration of the length of the mixing zone ($d^2\theta'/dt'^2$) which varies roughly as $(t')^{-0.4}$. Presumably in a sufficiently long cell a terminal velocity would be reached, but it is interesting to note that through rather advanced stages of nonlinear growth and for the wide range of viscosity contrasts reported herein, the mixing zone acceleration shows such a regular time dependence. The TA predictions for θ' vs t' are only qualitatively similar to this observation; their $A = 1$ calculation falls reasonably well on a power law with $\theta' \sim t'^{1.85}$ while their $A = 0$ calculation is essentially identical to the $A = 1$ at early times $t' < 25$ and subsequently shows a much slower growth of θ' ($\theta' \sim t'^{1.3}$). Not only is no such difference between $A \sim 1$ and $A \sim 0$ seen in this experiment but the $A \sim 1$ fingers are observed to develop at much earlier dimensionless time than are the $A \sim 0$ patterns.

A dimensionless stretching of the interface can be defined, following TA, as $L' = (L - L_0)/L_0$ where L is the length of the interface and L_0 is its initial value. For smooth fingers at very late times L'should vary linearly with θ', but this is not the case through the stages of pattern formation discussed here. Instead L' grows faster than θ'. In the $A \sim 0$ case this can be partly understood as a tendency for a few fingers to grow faster than others as the patterns first form but for the others to catch up at later times. These $A \sim 0$ cases all show a very similar dependence of L' on θ' which can be reasonably well expresed as $L' \sim (\theta')^{2.2}$. The $A \sim 1$ data show a less dramatic growth of L' with θ' along with a greater variation in this dependence, but they do appear to follow a power law with an exponent averaging ~ 1.6. This faster growth for L' than for θ' in the presence of stultification of some fingers and a secular reduction in the number of important fingers is very difficult to understand and appears to arise from an increase of complexity of the shape of the individual fingers. Nittman, Saccord, and Stanley[12] have reported observing fractal behavior in the shape of viscous fingers formed between two miscible liquids, one of which was non-Newtonian. It is possible (but not at all clear) that the interfacial stretching in the present case shows a last vestige of a fractal behavior which is smoothed by surface tension. Similarly the growth law $\theta' \sim (t')^{1.6}$ is reminiscent of the fractal growth calculated by Witten and Sanders[13] for diffusion-limited aggregation. Following a suggestion by Paterson,[3] Kandanoff and others of his group have been simulating the development of viscous fingering patterns with accumulations of random walks.[2]

Further analysis will concentrate upon distributions of curvature, fingure width, and finger length and their dependence on t', A, and B. However, the results presented in this paper show both dramatic differences in the development of nonlinear fingering patterns at different values of A and a strikingly simple power-law dependence on time of finger length for all values of the control parameters. Specifically these results are as follows: (1) neighboring fingers' growth rates do not affect each other noticeably for $A \sim 0$ but do for $A \sim 1$; (2) fingers do not amalgamate when $A \sim 0$ but they occasionally do when $A \sim 1$; and (3) the dimensionless interpenetration zone of the liquids follows the power law $\theta' \sim (t')^{1.6}$ at late times for all observed values of A and B.

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