Massive Gross-Neveu Model: A Rigorous Perturbative Construction

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We consider the massive Euclidean Gross-Neveu model. Thanks to the Pauli principle the bare perturbation expansion for the model with an ultraviolet cutoff is convergent in a disk whose radius corresponds by asymptotic freedom to a small finite renormalized coupling constant. The theory constructed in this way is physical (it satisfies Osterwalder and Schrader's axioms) in contrast with the planar theories obtained by similar perturbative expansions.

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Let us consider a fermionic theory in which the fermions have an explicit bare mass 1 (which fixes the energy scale). The fermions have N components and interact via the quartic Lagrangian $\mathscr{L}_I = \frac{1}{2}\lambda(\bar{\psi}\psi)^2$, $\lambda > 0$; we call this theory the massive Gross-Neveu model.[1] There does exist a Wick order on the interaction. For simplicity, we neglect it in the following. In contrast with bosonic theories, we remark that the partition function of such a theory in a finite box and with an ultraviolet cutoff has an absolutely convergent expansion no matter how large the (bare) coupling constant.

Using, as in Feldman *et al.*,[2] a multiscale cluster expansion, we can remove the constraint of a finite box, and obtain that the normalized perturbative expansion for the (infinite volume) Schwinger functions of the theory with fixed ultraviolet cutoff still converges for a bare coupling inside a small disk depending on the ultraviolet cutoff.

To remove the ultraviolet cutoff we consider the theory in two dimensions where it is renormalizable.[1] There are coupling-constant, mass, and wave-function renormalizations which, thanks to the asymptotic freedom of the model and to parity considerations, turn out to be finite. The sole source of divergence in the renormalized perturbative expansion is the appearance of renormalons (as in the four-dimensional planar model,[3–5] and in contrast with full bosonic theories like ϕ_4^4, which have also instantonlike singularities preventing the direct summation of the perturbation series).

Thanks to asymptotic freedom the renormalons do not prevent the construction of the theory. More precisely, the renormalized perturbation series (say in the Bogoliubov-Parisiuk-Hepp-Zimmermann scheme[6] of subtraction at zero external momenta) can be reshuffled into an unrenormalized series whose bare coupling shrinks with the ultraviolet cutoff at the same rate as the radius of convergence of the bare series.

In fact we work in the reverse way; starting from the correct *Ansatz* for the bare coupling λ_ρ as a function of the cutoff (which has to include the effect of the first two nonvanishing terms in the β function), we prove that the corresponding bare series converges for any cutoff, that the sum has a limit as the cutoff tends to infinity, and that the corresponding renormalized coupling is finite, nonzero (hence, the theory is not trivial). More precisely we can identify the theory constructed in this way with the Borel sum of the renormalized series (with renormalized finite mass and field strength).

Our main tool is the phase-space analysis which we used[2] to construct the infrared limit of massless ϕ_4^4; see also Gawȩdzki and Kupiainen.[7] However, the analysis here is simpler since there is no need to include a (nonperturbative) control of the "large field" regions responsible for the divergence of the bosonic perturbation expansions.

The model.—The bare mass is fixed to 1, and Λ is a compact box in R^2, while ρ is the ultraviolet cutoff index. The Lagrangian density is

$$\mathscr{L} = \bar{\psi}\,\partial\!\!\!/\,\psi + \tfrac{1}{2}(\bar{\psi}\psi) + \tfrac{1}{2}\lambda(\bar{\psi}\psi)^2,$$

where $\lambda > 0$. The fermionic propagator with exponential cutoff is (for some $M > 1$)

$$S_\rho(x-y) = \int \exp[ip(x-y)]\,\frac{p\!\!\!/+1}{p^2+1}\exp[-2p^2M^{-2(\rho+1)}]\,d^2p$$

where we used two-component spinors and

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad p\!\!\!/ = p_0\gamma_0 + p_1\gamma_1;$$

we have

$$\gamma_\alpha = -\gamma_\alpha^*, \quad \{\gamma_\alpha, \gamma_\beta\} = -2\delta_{\alpha\beta}.$$

It will be convenient to split S_ρ into two parts:

$$S_\rho(x-y) = \int A_\rho(x-t) B_\rho(t-y)\, d^2t,$$

where

$$\tilde{A}(p) = \frac{p+m}{(p^2+m^2)^{3/4}} \exp[-p^2 M^{-2(\rho+1)}], \quad \tilde{B}(p) = \frac{\exp[-p^2 M^{-2(\rho+1)}]}{(p^2+m^2)^{1/4}}.$$

Then the $2p$-point function $\mathscr{S}_{2p,\Lambda,\rho}$ for the theory with bare coupling constant λ_ρ is defined by

$$\mathscr{S}_{2p,\Lambda,\rho}(y_1,\ldots,y_p;z_1,\ldots,z_p) = S_{2p,\Lambda,\rho}(y_1,\ldots,y_p;z_1,\ldots,z_p)/Z_{\Lambda,\rho}$$

where Z is the normalization and

$$S_{2p,\Lambda,\rho}(y_1,\ldots,y_p;z_1,\ldots,z_p) = \sum_{n=0}^{\infty} \frac{(\lambda_\rho)^n}{n!} \int \psi(y_1)\cdots\psi(y_p)\bar{\psi}(z_1)\cdots\bar{\psi}(z_p)\Big[\int_\Lambda (\bar{\psi}\psi)^2(x)\, d^2x\Big]^n$$

$$= \sum_{n=0}^{\infty} \frac{(\lambda_\rho)^n}{n!} \int_\Lambda d^2x_1 \cdots d^2x_n \begin{pmatrix} y_1 \cdots y_p & x_1 x_1 \cdots x_n x_n \\ z_1 \cdots z_p & x_1 x_1 \cdots x_n x_n \end{pmatrix} \qquad (1)$$

where

$$\begin{pmatrix} u_i \\ v_j \end{pmatrix} \equiv \det[S_\rho(u_i - v_j)].$$

Using Gram's inequality and the symmetric form of the determinant in Eq. (1) one can bound $\binom{u_i}{v_j}$ by $\prod_i \| A(u_i,\cdot)\|_2 \| B(v_i,\cdot)\|_2$. Hence the radius of convergence of $S_{2p,\Lambda,\rho}$ as a power series in λ_ρ is infinite.

The convergence of the bare series.—To remove the cutoffs we split the propagator and thus the fields into momentum slices (the ith momentum slice corresponding to momenta of size M^i). In each slice i we perform a cluster expansion in squares of side M^{-i}. This is equivalent to a partial development of the determinant in Eq. (1). To preserve its symmetric form we apply the cluster expansion separately on each half A and B of the propagator. If we perform now a Mayer expansion, the normalized Schwinger function is a sum of products of polymers. The polymers are built out of squares; two squares of the same size can be joined by an explicit propagator; a square of size M^{-2i} contains all the ith frequency fields localized in it. This expansion has only a finite radius of convergence in λ_ρ, since to converge the Mayer expansion requires a small factor per square. In fact we prove the following estimate:

Theorem 1.—The radius of convergence r_ρ satisfies

$$r_\rho^{-1} \le -\beta_2 \ln M^\rho + (\beta_3/\beta_2)\ln\rho + K$$

for some numerical constant K, where β_2 and β_3 are the usual coefficients of the β function (in particular $\beta_2 < 0$).

The renormalization.—Let us start with $\lambda_\rho^{-1} = -\beta_2 \times \ln M^\rho + (\beta_3/\beta_2)\ln\rho + c$ with c large enough. We write $\lambda_\rho = \lambda_{\rho-1} + \delta\lambda_\rho$, where $\delta\lambda_\rho$ is the sum of the coupling-constant counterterms in the slice ρ. More generally,

$$\lambda_i = \lambda_{i-1} + \delta\lambda_i, \qquad (2)$$

where $\delta\lambda_i$ is the sum of the coupling-constant counterterms in the slices $\rho, \rho-1, \ldots, i$, which are not already in $\delta\lambda_\rho, \ldots, \delta\lambda_{i+1}$.

The above sums are absolutely convergent. We shall use the counterterms to renormalize the four-point subgraphs in the following way: We use the above relations to lower the index of the coupling constant of a vertex as long as it is strictly bigger than the index of its fields; we thus renormalize the four-point subgraphs whose internal lines are of higher momenta than the external fields hooked to it.

Since

$$\delta\lambda_i = \beta_2\lambda_i^2 \ln M + (\beta_3 - \beta_2^2 \ln M)\lambda_i^3 \ln M + O(\lambda_i^4) + O(e^{-i}),$$

one can prove that λ_i^{-1} behaves as $\beta_2 \ln M^i + (\beta_3/\beta_2)\ln i + c_i$ with c_i uniformly large and bounded. By power counting one can prove inductively that such an expansion is convergent (we follow the techniques of Feldman *et al.*,[8] and hence obtain Theorem 1. The estimates being uniform in ρ, the limit $\mathscr{S}$ of the Schwinger functions $\mathscr{S}_\rho$ as $\rho \to \infty$ exists. This limit satisfies obviously all Osterwalder-Schrader axioms by standard arguments.

Identification with the renormalized series of the Gross-Neveu model.—We can in the preceding development subtract and add the zero-momentum value of the two-point subgraphs and obtain a new development with renormalized two-point subgraphs and mass and wave function insertions. We obtain in this way a new propagator S_{ren}.

We invert the relations of Eq. (2) and obtain a development where $\lambda_\rho, \ldots, \lambda_1$ are functions of λ_0; the propagator is S_{ren} which we consider as independent of λ_0. Let λ be the value of the amputated truncated four-point function at zero momenta: $\lambda = \lambda_0 - \beta_2\lambda_0^2 + (\text{a convergent series in } \lambda_\rho, \ldots, \lambda_0)$. We invert this relation and consider now the theory as a function of λ (with propagator S_{ren} independent of λ).

Theorem 2.—$\mathscr{S}$ is a c^∞ function of λ and its Taylor series is the renormalized perturbation expansion of the Gross-Neveu model with propagator S_{ren}. Moreover for $\operatorname{Re}\lambda > 0$ and $|\lambda|$ small enough, S is an analytic function of λ and is Borel summable, i.e.,

$$|\mathscr{S} - \sum_{i=0}^{\infty} \alpha_i \lambda^i| \leq n![O(1)]^n \lambda^{n+1},$$

where α_i is the coefficient of λ^i in the Taylor series of $\mathscr{S}$.

To prove this we have just to complete for each vertex and each i the expansion of λ_i with Eq. (2), this time independently of the momenta of the fields of the vertex. To prove the Borel summability we stop the above procedure as soon as we have obtained all the terms of degree in λ less than or equal to n, and estimate the rest. In the rest we have produced n counterterms corresponding to four-point subgraphs whose internal momenta are smaller than the external momenta; each such counterterm is logarithmically divergent in the ultraviolet cutoff, thus logarithmic in the momenta of its fields. By the usual renormalon mechanism n logarithms give an $n!$.[8]

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