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Response to Parametric Modulation near an Instability

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The response of a damped anharmonic oscillator, being a model for many systems that undergo a bifurcation, is determined in the presence of small-amplitude parametric modulation of arbitrary dynamics and statistics. Inertia always stabilizes the trivial state and suppresses the growth of the bifurcating solution above the shifted stability threshold. The size of both effects depends on the modulation spectrum. Explicit formulas for threshold, bifurcating solution, and moments are given.

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Recently much research effort[1–9] has been devoted to investigate how nonlinear systems that upon quasistatic variation of a control parameter display an instability respond to a time-dependent modulation of the control parameter. In particular, Rayleigh-Bénard convection under periodic modulation has been investigated quite intensively.[1–4] The behavior of this system and of others which undergo under static conditions a "simple" bifurcation, e.g., a transition from a homogeneous state to a spatially structured state, is governed close to the instability fairly well by just one degree of freedom. That is, for example, the spatial Fourier mode of the hydrodynamic field that grows first. Furthermore, the dynamics of the amplitude $x(t)$ of this critical degree of freedom often corresponds[3] to the motion in a potential which changes from a single-minimum to a double-minimum shape when the control parameter crosses the critical value.

We therefore consider here a parametrically driven damped anharmonic oscillator,

$$m\ddot{x}(t)+m\gamma\dot{x}(t)=[\epsilon+\Delta\xi(t)]x(t)-x^3(t), \quad (1)$$

since it describes the response of many systems with a simple static bifurcation to an externally imposed modulation. Note that in many relaxation models there does appear a second-derivative inertia term when the time dependence of the driving is properly taken care of. The Ginzburg-Landau–type amplitude equations that have been derived from the fundamental field equations for static driving near, e.g., the convective instability are an example.

In the absence of modulation, $\Delta=0$, the system (1) shows an instability at the critical value $\epsilon_c(\Delta=0)=0$. For subcritical control parameters ϵ the trivial solution $x=0$ is stable while above threshold $x=\pm\sqrt{\epsilon}$ become stable. In this work we shall evaluate (i) the threshold value $\epsilon_c(\Delta)$ below which the trivial solution $x=0$ is stable and (ii) the nontrivial solution $\pm x(t)$ that bifurcates at $\epsilon_c(\Delta)$ in the presence of stationary small-amplitude modulation with zero mean, $\langle\xi(t)\rangle=0$, but otherwise arbitrary dynamics and statistics.[10] In particular, $\xi(t)$ can be periodic so that the modulation spectrum

$$D(\omega)=\int_{-\infty}^{\infty}dt\, e^{i\omega t}\langle\xi(t)\xi(0)\rangle \quad (2)$$

consists of sharp lines, or ξ can be a stochastic process with a broadband spectrum $D(\omega)$. In any case $D(\omega)$ is even, nonnegative for a real, stationary modulation process $\xi(t)$ with $\langle\xi(t)\xi(t')\rangle=D(|t-t'|)$. The above angular brackets denote appropriate time averages or in the case of stochastic driving ensemble averages with the statistical weight given by the path probability distribution of the process ξ.

For small driving amplitudes Δ the time-averaged behavior of the system turns out to be similar to that in the static case, $\Delta=0$, with no modulation as one might have expected: The mean squared amplitude, for example, grows linearly with the supercritical distance of the control parameter ϵ from threshold $\epsilon_c(\Delta)$,

$$\langle x^2(t)\rangle = [\epsilon-\epsilon_c(\Delta)]s_2(\Delta)+O((\epsilon-\epsilon_c)^2), \quad (3)$$

and vanishes below. The size of the slope $s_2(\Delta)$ and of $\epsilon_c(\Delta)$ depends functionally on the modulation statistics and dynamics. However, the stability domain of the trivial state $x=0$ is always enlarged to positive control parameters, $0=\epsilon_c(\Delta=0)\leqslant\epsilon\leqslant\epsilon_c(\Delta)$. Furthermore, the growth of the mean squared amplitude is *always* suppressed, $s_2(\Delta)\leqslant s_2(\Delta=0)=1$ as shown schematically in Fig. 1.

Large-amplitude modulation, on the other hand, may cause stabilization or destabilization and the response can be arbitrarily complicated even close to a stability boundary of the $x=0$ solution. We therefore investigate the behavior of the system for small modulation amplitudes Δ and control parameters ϵ close to the threshold $\epsilon_c(\Delta)$ being itself close to $\epsilon_c(\Delta=0)=0$. In such situations the potential configurations shown schematically in Fig. 2 will typically occur in the course of the modulation.

To determine the bifurcating solution $x(t)$ and the threshold $\epsilon_c(\Delta)$ we use the Poincaré-Lindstedt expansion

$$x(t)=\sum_{n=1}\lambda^n x_n(t), \quad \epsilon-\epsilon_c=\sum_{n=1}\lambda^n\epsilon_n \ , \quad (4)$$

assuming that x grows continuously with the distance $\epsilon-\epsilon_c$ from threshold which seems to be true for small Δ. The expansion parameter λ measuring the size of the root mean square of x may then be eliminated in favor of $(\epsilon-\epsilon_c)^{1/2}$ in our case.

With (4) the nonlinear problem (1) is transformed into a sequence of linear equations. The first one,

$$\mathscr{L}x_1=[m\,\partial_t^2+m\gamma\,\partial_t-\epsilon_c(\Delta)-\Delta\xi(t)]x_1=0, \quad (5)$$

is just (1) linearized around the trivial solution at its stability boundary. The expansion (4) makes sense only if the solution x_1 of the linearized problem at threshold is bounded. From the Fredholm solvability condition $\langle[\mathscr{L}^+y_1]x_n\rangle=\langle y_1\mathscr{L}x_n\rangle=0$ for $n=2$ and $n=3$ one finds

$$\epsilon_1=0, \quad \epsilon_2=\langle y_1x_1^3\rangle/\langle y_1x_1\rangle, \quad (6)$$

where $y_1(t)$ solves the linear problem adjoint to (5),

$$\mathscr{L}^+y_1=[m\,\partial_t^2-m\gamma\,\partial_t-\epsilon_c(\Delta)-\Delta\xi(t)]y_1=0. \quad (7)$$

To lowest order the bifurcating solution above threshold is determined by x_1 and y_1,

$$x(t)=[(\epsilon-\epsilon_c)\langle y_1x_1\rangle/\langle y_1x_1^3\rangle]^{1/2}x_1(t) + O(\epsilon-\epsilon_c). \quad (8a)$$

Its equal-time moments grow like

$$\langle x^n\rangle=[\epsilon-\epsilon_c(\Delta)]^{n/2}s_n(\Delta) + O((\epsilon-\epsilon_c)^{(n+2)/2}), \quad (8b)$$

$$s_n(\Delta)=\langle x_1^n\rangle\langle x_1y_1\rangle^{n/2}\langle y_1x_1^3\rangle^{-n/2}. \quad (8c)$$

Here all arguments have been suppressed in the above equal-time averages. Note that the nonlinearity in (1) enters via (6).

Still we have to determine $\epsilon_c(\Delta)$ and the functions $x_1(t;\Delta)$ and $y_1(t;\Delta)$ solving (5) and (7), respectively. This is done for small Δ via a second expansion[4]

$$\epsilon_c(\Delta)=\sum_{n=0}\Delta^n\epsilon_c^{(n)}, \quad x_1(t;\Delta)=\sum_{n=0}\Delta^n x_1^{(n)}(t), \quad (9)$$

and similarly for $y_1(t;\Delta)$. Then Eqs. (5) and (7) are decomposed into a sequence of linear second-order differential equations with constant coefficients. Here $\epsilon_c^{(0)}=0$ while $x_1^{(0)}$ and $y_1^{(0)}$ are undetermined constants. Since they drop out in observables like (8) we may set $x_1^{(0)}=y_1^{(0)}=1$. Then the next two orders

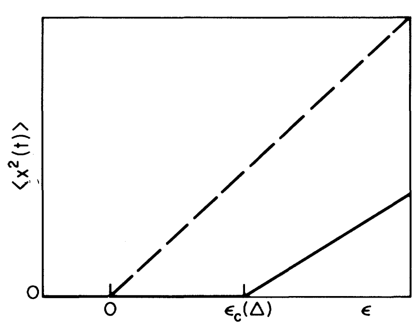

FIG. 1. Schematic bifurcation diagram of the mean squared amplitude $\langle x^2\rangle$ as a function of control parameter in the presence of modulation with small amplitudes Δ (solid line) and without modulation, $\Delta=0$ (dashed line).

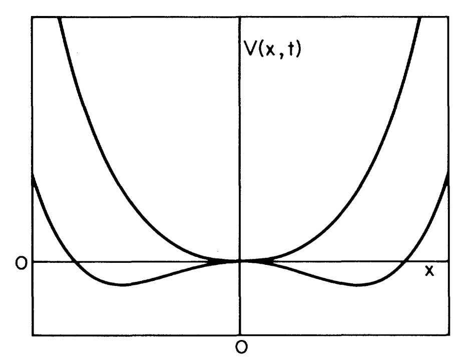

FIG. 2. Typical configurations of the potential $V(x,t)=-[\epsilon+\Delta\xi(t)]x^2/2+x^4/4$ occurring for small-amplitude modulation close to $\epsilon=0$.

read ($\mathscr{L}^{(0)} = m\,\partial_t[\partial_t+\gamma]$)

$$\mathscr{L}^{(0)} x_1^{(1)}(t) = \epsilon_c^{(1)} + \xi(t), \quad \epsilon_c^{(1)} = 0, \tag{10a}$$

$$\mathscr{L}^{(0)} x_1^{(2)}(t) = \epsilon_c^{(2)} + x_1^{(1)}(t)\xi(t),$$
$$\epsilon_c^{(2)} = -\langle x_1^{(1)}(t)\xi(t)\rangle, \tag{10b}$$

with $\epsilon_c^{(1)}$ and $\epsilon_c^{(2)}$ following from the Fredholm alternative.

Thus we arrive at the first nontrivial result: The threshold shift

$$\epsilon_c(\Delta) = \Delta^2 \frac{1}{m}\int_0^\infty \frac{d\omega}{\pi}\frac{D(\omega)}{\omega^2+\gamma^2} + O(\Delta^4) \tag{11}$$

is positive. *Any* small-amplitude modulation stabilizes the trivial solution $x=0$, i.e., the basic, spatially homogeneous state in the hydrodynamical system represented by (1). Its stability range is extended[11] to a driving domain, $0<\epsilon\leq\epsilon_c(\Delta)$, in which the time-averaged potential has a maximum at the origin (cf. the dynamical stabilization of a pendulum in the statically unstable upward position). Each frequency mode ω of the modulation contributes additively to the stabilization. Its weight is given by the spectral intensity $D(\omega)$ and a dynamical factor which is determined by the response function of the system and which monotonously drops from $1/m\gamma^2$ at low frequencies to zero at high frequencies. To second order in Δ the threshold shift $\epsilon_c(\Delta)$ is determined solely by the two-point correlation spectrum of the modulation. A four-point correlation function of ξ enters into the next nonvanishing order (which is Δ^4 for symmetric distributions of the modulation amplitudes around zero). Thus the threshold shift (11) is independent of the statistics of the modulation.

The dynamical stabilization by small-amplitude modulation is caused by inertia. It hinders the particle's escape from the fixed point $x=\dot{x}=0$ when the potential is varied as shown in Fig. 2. The threshold shift vanishes linearly with $m\to 0$ in the overdamped case, $m\gamma\to 1$, of a pure relaxation dynamics. For such a dynamics the mean squared amplitude is given by $\langle x^2(t)\rangle = \epsilon$ above the *unshifted* threshold $\epsilon_c = 0$. Our expansion reproduces this exact result which follows from taking the average of the equation of motion $\dot{x}/x = \epsilon + \Delta\xi - x^2$ above threshold where $x(t)$ is nonzero (if x becomes zero it remains so since $x=0$ implies *here* $\dot{x}=0$). Since other moments of the stationary probability distribution are finite for $\epsilon>0$ as well, given by $\langle x^{2n}\rangle = \epsilon^n s_n(\Delta) + O(\epsilon^{n+1})$, only the most probable value of x can be stabilized by modulation[7] in a relaxational system.

The formula (11) holds for arbitrary small-amplitude modulation. It contains the stability boundary derived from the Mathieu equation for $\xi(t) = \cos(\Omega t)$, i.e., $D(\omega) = [\delta(\omega-\Omega)+\delta(\omega+\Omega)]\pi/2$, consisting of a line spectrum.[3] It also contains the threshold that Graham and Schenzle derived with a different technique for a Gaussian stochastic process with a Brownian spectrum.[8] For such a driving these authors have also given formulas for the moments which for $n=2$ can be cast into the form (8b), however, with $s_2(\Delta=0)=1$.

As a second nontrivial result we not only obtain the moments but also the bifurcating solution $x(t)$ itself shortly above threshold by inserting the small-Δ expansion for x_1 and y_1 into (8). The growth coefficients $s_n(\Delta)$ (8c), e.g., are

$$s_n(\Delta) = 1 - n\Delta^2\langle x_1^{(1)} y_1^{(1)} + (2-n/2)(x_1^{(1)})^2\rangle + O(\Delta^4). \tag{12}$$

In particular, the initial slope

$$s_2(\Delta) = 1 - \Delta^2 \frac{4}{m^2}\int_0^\infty \frac{d\omega}{\pi}\frac{D(\omega)}{(\omega^2+\gamma^2)^2} O(\Delta^4) \tag{13}$$

of the mean squared amplitude is in the presence of modulation and inertia always smaller (cf. Fig. 1) than for static driving: Small-amplitude modulation not only delays the appearance of the nontrivial state but it also suppresses its growth above threshold.

We mention that the Δ expansion exhibits infrared divergencies if the modulation has a finite zero-frequency Fourier component. The reason for the growth of, e.g., $x_1^{(1)}(t)$ is the shift of the control parameter ϵ away from threshold induced by the additional static driving $\Delta\xi(\omega=0)$. The infrared problem[3,12] is most obvious for a monochromatic modulation $\xi(t)=\cos(\Omega t)$ for which $x_1^{(1)} \sim \Omega^{-1}\mathrm{Re}[(\Omega + i\gamma)^{-1}e^{-i\Omega t}]$. An *ad hoc* remedy is to exclude zero-frequency modulation by the restriction $D(\omega=0)=0$ in the lowest-order Δ expansion and similar restrictions in higher orders. The expressions (11) and (13) for the observables $\epsilon_c(\Delta)$ and $s_2(\Delta)$, however, do not require such a restriction.

We hope that our predictions for the threshold shift, the bifurcating solution, and its moments will be compared soon with experiments. Nonlinear electrical circuits, mechanical instabilities, and systems which can be reduced to (1) would be most appropriate. However, the method presented here can also be applied to modulated systems more complicated than (1) to predict *quantitatively* the threshold and the behavior shortly above.[13]

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[10]Since Eq. (1) admits the $x=0$ solution for arbitrary driving the bifurcation(s) at the threshold(s) $\epsilon_c(\Delta)$ where $x=0$ becomes unstable remain sharp also in the driven system: With *parametric* driving coupling *multiplicatively* to the order parameter x there is no rounding or smearing of the transition(s) from $x=0$ to finite $x(t)$ as in the presence of *additive* forcing.

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[13]Recently it has been applied (Ref. 3) successfully to determine the response of a Rayleigh-Bénard system to external sinusoidal temperature and gravity modulation.