Equation of Motion of a Stringlike Dislocation

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A differential equation is proposed to describe the evolution of a screw dislocation in a continuous medium under an externally applied displacement field. The derivation is based on the conservation of energy and momentum and the neglect of radiation reaction and renormalizability: The equation of motion must be such that it allows for a renormalization of the infinite self-energy and momentum of the infinitely thin dislocation. The extent to which the result might hold for general dislocation loops is discussed.

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An old question[1] in the theory of dislocation dynamics in an elastic continuum is the following: Is it possible to determine the motion of a dislocation in a prescribed stress field? How to find the displacement field generated by a dislocation loop in a given arbitrary motion is well known[2]; it is, by use of an electrodynamic analogy, similar to solving Maxwell's equations for the fields given the sources. The motion of the sources themselves, i.e., point-charged particles, under prescribed external fields is given, of course, by $F=ma$ where F is the Lorentz force if radiation reaction is neglected. This equation was modified in a Lorentz-invariant way by Dirac[3] in order to include radiation reaction, and he showed that the divergent self-energy of a point particle could be regularized through a mass renormalization. His use of advanced fields was shown to be unnecessary by Teitelboim,[4] and a number of ambituities in the renormalization procedure where later removed by Tabensky.[5] It had also been previously emphasized by Feynman[6] that regularization of the action of a charged particle on itself meant that, at the classical level, the mass of the particle could be considered as purely electromagnetic.[7] Feynman's approach was used by Lund and Regge[8] to show that in a theory of Nambu strings interacting via an antisymmetric tensor field[9] the action of a string on itself could be regularized through a renormalization of the slope of the Regge trajectories, alias the string tension.

Take now an infinite, three-dimensional, homogeneous, linearly elastic continuum of density ρ and elastic constant tensor[10] c_{ijkl}. The dynamical variable is a field $\mathbf{u}(\mathbf{y},t)$ representing the (small) displacement at time t of a particle from its equilibrium position $\mathbf{y}$. A string-like dislocation loop is an infinite or closed curve $\mathbf{X}(\sigma,t)$, σ being a parameter, with Burgers vector $\mathbf{b}$. The field $\mathbf{u}$ is multivalued but its derivatives are single valued and so are the energy and momentum of the field which are, however, singular along the loop. The idea is to think of the loop $\mathbf{X}(\sigma,t)$ as the source of the elastodynamic field $\mathbf{u}$ in the same sense that point-charged particles are the source of the electromagnetic field, and to take them seriously as dynamical objects in their own right.

A convenient form for the velocity and strain fields generated by a dislocation loop in arbitrary motion was found by Mura[11]

$$\partial u_m/\partial t=\int d^3x'\,dt'\,G_{mk}(\mathbf{x}-\mathbf{x}';t-t')f_k(\mathbf{x}',t'), \quad (1)$$

$$\partial u_m/\partial x^n =\int d^3x'\,dt'\,G_{mk}(\mathbf{x}-\mathbf{x}';t-t')g_{kn}(\mathbf{x}',t'), \quad (2)$$

where G is the elastodynamic Green's function and f_k and g_{kn} are functionals of $\mathbf{X}(\sigma,t)$ that vanish everywhere except along the loop itself. These formulas are remarkable as the slip plane does not appear, and they are analogous to the Lienard-Wichert fields of a moving point charge. They give a precise meaning to the statement that the dislocation loop acts as a source of an elastodynamic field.

As was already said, strain and velocity given by (1) and (2) are singular along the loop. At short distances from it they are finite but large, and this means that linear elasticity is no longer adequate and nonlinear effects have to be considered; moreover, for distances comparable to the interatomic spacing the continuum approximation breaks down and atomic structure has to be taken into account. The following question then comes to mind: Is it possible to sweep both problems under some rug and retain a description of dislocations as infinitely thin strings whose motion in an external stress field is determined, say, by a differential equation of evolution? The answer is yes, with the assumption that the energy radiated by the dislocation as it moves is negligible compared with the work done by the external stresses upon the dislocation. It is possible to endow stringlike dislocations with mass and string tension of purely elastic origin at least, at the level that is reported here, when radiation reaction is not considered. This means that results derived from the theory will be valid as long as accelerations are not too large. On the other hand, insisting on a stringlike description of dislocations gives a scale-free theory that should be applicable in widely different contexts such as internal friction,[12] fracture at the laboratory[13] and geophysical[14] scales, phase transitions,[15] and any situation where the relevant wavelengths are long compared with atomic distances.

The starting point of computations is the action integral whose extrema give the equations of classical elasticity:

$$S=\int dt\, d^3x\left[\frac{\rho}{2}\left(\frac{\partial \mathbf{u}}{\partial t}\right)^2+\frac{1}{2}c_{ijkl}\frac{\partial u^i}{\partial x^j}\frac{\partial u^k}{\partial x^l}\right].$$

Invariance under space and time translation imply the existence of a locally conserved energy-momentum tensor:

$$\frac{\partial T^{00}}{\partial t}+\frac{\partial T^{0j}}{\partial x^j}=0,\quad \frac{\partial T^{i0}}{\partial t}+\frac{\partial T^{ij}}{\partial x^j}=0, \tag{3}$$

where the index zero means time component. Consequently, the energy E and momentum P^i of the field,

$$E=\int d^3x\, T^{00},\quad P^i=\int d^3x\, T^{i0}, \tag{4}$$

are constants of motion.

The total displacement due to a dislocation loop and externally applied stresses in a linear medium will be the sum $\mathbf{u}+\mathbf{U}$ of the displacements that each factor would contribute if the other were absent. Correspondingly, the energy-momentum tensor will be split into three:

$$T=T_s+T_m+T_e, \tag{5}$$

where T_s (for self) involves only $\mathbf{u}$, T_m (for mixed) involves products of $\mathbf{u}$ and $\mathbf{U}$, and T_e (for external) involves only $\mathbf{U}$. Velocities and strains generated by the external loads are supposed to be regular everywhere, and T_e will satisfy (3) identically. T_s and T_m, however, are singular at the dislocation loop and the conservation laws (3) have to be modified. This is done as follows: surround the dislocation loop by a very thin tube; outside this tube (3) will hold and energy and momentum will be defined by (4). The self-energy E_s and momentum P_s^i will be very large for small tube thickness. So, *define* the energy and momentum inside the tube in such a way that its sum with E_s and P_s^i is finite. This sum will be called the energy and momentum of the stringlike dislocation, it will be well defined everywhere, and the conservation laws will provide equations to find out how they change under the influence of an external field. In this way dislocations are given an existence independent of the elastodynamic field.

In the sequel the case of a screw dislocation is considered. The algebra is thereby greatly simplified while retaining most of the relevant physics.[16] In this case

$$\mathbf{X}(\sigma,\tau)=(X_1(\tau),X_2(\tau),\sigma),$$
$$\mathbf{b}=(0,0,b),$$

the problem is one of antiplane strain, only the third component of displacement is coupled to the dislocation, and it is enough to take

$$\mathbf{u}=(0,0,u),\quad \partial u/\partial z=0.$$

This simplifies (1) and (2) to

$$\begin{aligned}\partial u/\partial y^a&=b\mu\int d\tau\, d\sigma\,\epsilon_{ab}\partial G/\partial y^b+b\rho\int d\tau\, d\sigma\,\epsilon_{ab}(\partial X_b/\partial\tau)(\partial G/\partial t),\\ \partial u/\partial t&=b\mu\int d\tau\, d\sigma\,\epsilon_{ab}(\partial X_b/\partial\tau)(\partial G/\partial y^a),\end{aligned} \tag{6}$$

with $\epsilon_{12}=-\epsilon_{21}=1$, $\epsilon_{11}=\epsilon_{22}=0$,

$$4\pi\mu G(t-\tau;\mathbf{y}-\mathbf{X})=\frac{\delta(t-\tau-\beta^{-1}|\mathbf{y}-\mathbf{x}|)}{|\mathbf{y}-\mathbf{X}|},$$

where μ is the shear modulus and β the shear wave velocity.

According to the procedure outlined above, this dislocation is now surrounded by a thin tube. From (3) and (4) one has

$$\frac{d}{dt}(\text{momentum outside tube})=\int d^2S_b(T^{ab}-V^bT^{a0}),\quad \frac{d}{dt}(\text{energy outside tube})=\int d^2S_b(T^{0b}-V^bT^{00}), \tag{7}$$

where d^2S_b is the surface element of the tube and V^b is the velocity of the dislocation at time t. To evaluate (7) the fields (6) are needed at close distance from the dislocation, say at $y^a=X^a(t)+\epsilon^a$ with ϵ^a small. Substitution into (6) yields[17]

$$\begin{aligned}\frac{\partial u}{\partial y^a}&=-\frac{b\beta^2\gamma\epsilon_{ab}\epsilon^b}{2\pi E^2}+\frac{b\beta\epsilon_{ab}A^b}{4\pi\beta^3\gamma^3}\ln\frac{\beta\delta}{E}+(\text{finite terms}),\\ \frac{\partial u}{\partial t}&=\frac{b}{4\pi\gamma E^2}V^a\epsilon_{ab}\frac{\partial E^2}{\partial\epsilon^b}-\frac{bV^a\epsilon_{ab}A^b}{4\pi\beta^2\gamma^3}\ln\frac{\beta\delta}{E}+(\text{finite terms}),\end{aligned} \tag{8}$$

where A^a is the acceleration at time t, $\gamma^2 = 1 - V^2/\beta^2$, δ is a cutoff implementing the requirement that radiation emitted from one portion of the dislocation does not affect the motion of another region, and

$$E^2 = (V \cdot E)^2 + (\beta^2 - V)^2 \epsilon^2.$$

The leading term in strain and velocity for short distances diverges like an inverse power, while the next one diverges logarithmically. It is not unlikely that this type of short-distance behavior will also hold for more general dislocation loops.

The tube will now be taken as a cylinder of elliptical cross section[18] $E^2 = \text{const}$. Substitution of (8) into (7) gives the rate of change of energy and momentum per unit length outside the tube:

$$\begin{aligned} \frac{dE_m}{dt} &= -\mu b \epsilon_{ab} V^a \frac{\partial U}{\partial y^b}, \\ \frac{dP_m^a}{dt} &= -\mu b \epsilon_{ac} \frac{\partial U}{\partial y^c} - \rho b \epsilon_{ac} V^c \frac{\partial U}{\partial t}, \\ \frac{dE_s}{dt} &= \frac{\mu b^2}{4\pi} \ln \frac{\delta\beta}{E} \frac{d\gamma^{-1}}{dt}, \\ \frac{dP_s^a}{dT} &= \frac{\mu b^2}{4\pi\beta^2} \ln \frac{\delta\beta}{E} \frac{d(V^a \gamma^{-1})}{dt}, \end{aligned} \tag{9}$$

The self-energy and momentum diverge logarithmically when $E \to 0$. Now, *postulate* that dislocations are objects (i.e., strings) endowed with energy and momentum proportional to γ^{-1} and $\gamma^{-1} V^a$, respectively. This makes it possible to write the following equation of motion for a screw dislocation:

$$dP^a/dt = F^a, \tag{10}$$

in which $P^a = MV^a\gamma^{-1}$ is the momentum of the dislocation with M being a parameter with dimension of mass per unit length to be determined by experiment, and the external force is

$$F^a = \mu b \epsilon_{ac} (\partial U/\partial y^c) + \rho b \epsilon_{ac} V^c (\partial U/\partial t).$$

The first term on the right is the usual[19] Peack-Hoehler force, while the second was guessed by Eshelby[20] on the basis of an analogy with lines of constant electric charge. Similarly, the energy equation becomes

$$dE/dt = F \cdot V, \tag{11}$$

with $E = M\beta^2\gamma^{-1}$, the energy of the dislocation, meaning that the rate of energy change for the dislocation is equal to the rate at which work is done on it by external forces. These are the same equations obeyed by a relativistic line of uniform charge.

Expression (10) is the announced evolution equation: It involves a mass M of purely elastic origin which hides all the nonlinear behavior at the core. The renormalization employed means that M should be the same for all dislocations with the same Burgers vector. Technically, the derivation is made possible by the short distance behavior (9) which depends on the locality of the Green's function G. This locality holds, at short distances, not only for antiplane but also for in-plane and three-dimensional problems. Thus, equations similar to (10) should be obtainable for edge dislocations and for arbitrary dislocation loops. Work along this line is in progress.

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