Conformal Invariance and the Yang-Lee Edge Singularity in Two Dimensions

John L. Cardy
Department of Physics, University of California, Santa Barbara, California 93106
(Received 8 January 1985)

It is shown that very general features of the critical theory of the Yang-Lee edge singularity in two dimensions completely determine the way in which the theory realizes conformal invariance. This leads to the value $\sigma = -\frac{1}{6}$ for the edge exponent, and makes possible the calculation of the correlation functions.

PACS numbers: 05.50.+q, 05.70.Jk

Recently there has been considerable progress in exploiting the principle of conformal invariance of two-dimensional systems at the critical point to obtain information about critical exponents and correlation functions.[1,2] Unfortunately this principle by itself is not sufficiently restrictive, and other criteria, such as full unitarity of the theory,[2] have been invoked. While such criteria are necessary for a sensible quantum field theory, many interesting critical points do not correspond to unitary theories. Another difficulty of this approach is in the identification of a given realization of conformal symmetry with a particular universality class. So far, this has been accomplished only by matching the predicted exponents, or the conformal anomaly c, with those values already known by other means.

The Yang-Lee edge singularity[3,4] is perhaps the simplest nonunitary critical point. In addition, as will be discussed below, it also corresponds to the simplest universality class. It will turn out that these properties are sufficient to determine a simple way in which conformal invariance can be realized in this model in two dimensions. This determines the critical exponents and the correlation functions.

The Yang-Lee edge singularity[3,4] occurs in an Ising model above its critical temperature in a nonzero, purely imaginary magnetic field ih. For h larger than some critical value $h_c(T)$ the partition function acquires zeros, which become dense on the line $\mathrm{Re}h > h_c$ in the thermodynamic limit. The density of these zeros behaves near h_c like[5] $(h-h_c)^\sigma$. Fisher[6] showed how the point $h = h_c$ can be regarded as a conventional critical point. In high dimensions it corresponds to the infrared behavior of the field theory of a single scalar field $\phi(\mathbf{r})$ with an action

$$A = \int d^d r \, [\tfrac{1}{2}(\nabla\phi)^2 + i(h-h_c)\phi + \tfrac{1}{3}ig\phi^3]. \qquad (1)$$

The imaginary coupling makes the theory nonunitary. The critical point is where the renormalized coefficients of ϕ and ϕ^2 vanish. In $6-\epsilon$ dimensions there are apparently two relevant fields, coupling to ϕ and ϕ^2. However, correlations of ϕ^2 are related to those of ϕ by the equation of motion, so in fact ϕ^2 is a redundant operator.[7] The two-point function $\langle\phi(\mathbf{r}_1)\phi(\mathbf{r}_2)\rangle$ behaves like $|\mathbf{r}_1-\mathbf{r}_2|^{-2x}$ at the critical point, where $2x = d-2+\eta$ is related to σ by Fisher's relation[6] $\sigma = (d-2+\eta)/(d+2-\eta)$. The simplicity of this universality class lies in its lack of any internal symmetry, and in the existence of only one independent (relevant) exponent.

In order to characterize the theory in two dimensions, the following properties [valid to all orders in the $(6-d)$ expansion[8]] will be assumed: (a) there is only one[9] relevant operator ϕ; (b) the three-point function $\langle\phi(\mathbf{r}_1)\phi(\mathbf{r}_2)\phi(\mathbf{r}_3)\rangle$ is nonzero.

Belavin, Polyakov, and Zamolodchikov[1] have shown that there is a large class of field theories which realize conformal symmetry in a simple way. The allowed scaling dimensions of scalar operators in these theories are given by the Kac formula[10]

$$x_{p,q} = \tfrac{1}{2}[(p\alpha_+ + q\alpha_-)^2 - (\alpha_+ + \alpha_-)^2], \qquad (2)$$

where p,q are positive integers, $\alpha_\pm = \alpha_0 \pm (1+\alpha_0^2)^{1/2}$, and α_0 is related to the conformal anomaly c of the theory by $c = 1-24\alpha_0^2$. In such theories, the correlation functions satisfy linear differential equations, and there are restrictions on which three-point functions may be nonzero: $\langle\phi_{p_1,q_1}\phi_{p_2,q_2}\phi_{p_3,q_3}\rangle$ is zero unless (i) $p_1+p_2+p_3 = 1$ (mod2), (ii) (p_1-1), (p_2-1), and (p_3-1) satisfy the triangle inequalities $(p_1-1)+(p_2-1) \leq (p_3-1)$, etc. Similar conditions must be satisfied by the q_i. For theories where α_+/α_- is a rational number, these conditions imply that there are only a finite number of basic operators in the theory. Such cases appear to be connected to integrable models in a way which is not yet understood.[11]

Let us assume that the critical theory corresponding to the Yang-Lee edge in two dimensions is in the class considered by Belavin, Polyakov, and Zamolodchikov, and use conditions (a) and (b) as constraints on the possible realizations. Since ϕ is to be relevant, it must have a scaling dimension $x < 2$. This restricts it to lie in the strip

$$\alpha_- - \alpha_+ < p\alpha_+ + q\alpha_- < \alpha_+ - \alpha_-. \qquad (3)$$

The conditions (i) and (ii) imply that the correlation functions $\langle\phi_{p,q}\phi_{p,q}\phi_{p',q'}\rangle$ may be nonzero if $p' = p \pm 1$

(p even) or $p'=p$, $p\pm2$ (p odd), and similarly for q'. If p and q satisfy (3) it is easy to see graphically that in general at least two of the possibilities for (p',q') also satisfy (3), so that $\phi_{p,q}^2$ will, in general, couple to other relevant operators.[12] Thus in general it is very difficult to satisfy condition (a). The only possibility is that $\phi_{p,q}^2$ is allowed to couple to just two operators, one of which must be the unit operator $(p',q')=(1,1)$, the other having a dimension $x_{p',q'}$ equal to that of $\phi_{p,q}$, so they are physically the same operator. In this way, condition (b) is also satisfied. This will occur if $(p,q)=(1,2)$ [or (2,1), which is equivalent]. In the former case the only nonzero three-point function is $\langle\phi_{1,2}\phi_{1,2}\phi_{1,3}\rangle$, where $x_{1,2}=x_{1,3}$. Using the Kac formula (2), after some simple algebra one obtains[13] $x=-\frac{2}{5}$ and $\alpha_0=3/(40)^{1/2}$, corresponding to $c=-\frac{22}{5}$. Fisher's scaling relation[6] then gives $\sigma=-\frac{1}{6}$.

Since ϕ corresponds to $(p,q)=(1,2)$, it is degenerate on level 2, and it follows from the work of Belavin, Polyakov, and Zamolodchikov[1] that all the correlation functions satisfy second-order linear differential equations. The calculation of the four-point function closely parallels that of Dotsenko[11] for the three-state Potts model. The result is[14]

$$\langle\phi(z_1)\phi(z_2)\phi(z_3)\phi(z_4)\rangle = \left|\frac{z_{13}z_{24}}{z_{12}z_{23}z_{34}z_{14}}\right|^{-4/5}\{|F_1(\zeta)|^2+C|F_2(\zeta)|^2\},$$

where $\zeta=z_{12}z_{34}/z_{13}z_{24}$, and

$$F_1(\zeta)={}_2F_1(\tfrac{3}{5},\tfrac{4}{5};\tfrac{6}{5};\zeta),$$

$$F_2(\zeta)=\zeta^{-1/5}{}_2F_1(\tfrac{3}{5},\tfrac{2}{5};\tfrac{4}{5};\zeta),$$

$$C=-\frac{\Gamma(\frac{6}{5})^2\Gamma(\frac{1}{5})\Gamma(\frac{2}{5})}{\Gamma(\frac{3}{5})\Gamma(\frac{4}{5})^3}.$$

By considering the limit $z_{13}\to\infty$ with z_{12},z_{34} fixed one may extract the normalized three-point function[15]

$$\langle\phi(z_1)\phi(z_2)\phi(z_3)\rangle = i|C|^{1/2}|z_{12}z_{23}z_{31}|^{2/5}.$$

Note that this is purely imaginary as required by the Lagrangian (1).

Finally, the result $\sigma=-\frac{1}{6}$ may be compared with other numerical and exact information. Series analysis[16] leads to the value $\sigma=-0.163\pm0.003$, and the ϵ expansion, suitably resummed, gives[8] $\sigma=-0.155\pm0.01$. (The quoted error reflects the spread in different extrapolations.) It has also been argued[17] that σ is related to the exponent θ of directed lattice animals in three dimensions by $\theta=1+\sigma$. A particular model of three-dimensional directed animals has been solved by Dhar,[18] by mapping the problem onto the unphysical critical point of Baxter's hard hexagon model.[19] He finds $\theta=\frac{5}{6}$, consistent with $\sigma=-\frac{1}{6}$. In that work, a correction-to-scaling exponent $\Omega=\frac{5}{6}$ was also found. This corresponds to an operator with a renormalization-group eigenvalue $y=-\Omega/\nu_\perp$. Since $2\nu_\perp=\theta$,[17] it follows that $y=-2$. This value is consistent with the result of the present paper that the operator algebra closes with only one operator ϕ, since the conformal symmetry does not rule out the possibility of other scaling fields with negative integral eigenvalues.

In conclusion, the critical theory of the Yang-Lee edge corresponds to a rather simple, albeit nonunitary, realization of conformal symmetry. This suggests that it corresponds to an exactly solvable model.

The author thanks A. Luther and B. McCoy for comments. This work was supported by the National Science Foundation under Grant No. PHY83-13324.

---

[1]A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, J. Stat. Phys. **34**, 763 (1984); Nucl. Phys. **B241**, 333 (1984).

[2]D. Friedan, Z. Qiu, and S. Shenker, Phys. Rev. Lett. **52**, 1575 (1984).

[3]C. N. Yang and T. D. Lee, Phys. Rev. **87**, 404 (1952).

[4]T. D. Lee and C. N. Yang, Phys. Rev. **87**, 410 (1952).

[5]P. J. Kortman and R. B. Griffiths, Phys. Rev. Lett. **27**, 1439 (1971).

[6]M. E. Fisher, Phys. Rev. Lett. **40**, 1610 (1978).

[7]See, for example, F. Wegner, in *Phase Transitions and Critical Phenomena, Vol. 6*, edited by C. Domb and M. S. Green (Academic, New York, 1976), p. 34.

[8]Numerical extrapolations of the ϵ expansion appear to be remarkably accurate even at $d=2$. See O. de Alcantara Bonfim, J. E. Kirkham, and A. J. McKane, J. Phys. A **14**, 2391 (1981).

[9]Technically, this means that the operator product expansion of $\phi(\mathbf{r}_1)\phi(\mathbf{r}_2)$ generates no other relevant operators. This is equivalent to requiring that the only nontrivial three-point function of relevant operators is $\langle\phi\phi\phi\rangle$.

[10]V. G. Kac, in *Group Theoretical Methods in Physics*, edited by W. Beiglbock and A. Bohm, Lecture Notes in Physics, Vol. 94, (Springer-Verlag, New York, 1979), p. 441; B. Feigin and D. B. Fuchs, Funkts. Anal. Prilozhen **16**, 47 (1982) [Funct. Anal. Appl. **16**, 114 (1982)].

[11]Vl. S. Dotsenko, Nucl. Phys. B **235** [FS11], 54 (1984).

[12]Since conditions (i) and (ii) give a necessary condition only, it is possible that $\phi_{p',q'}$ decouples for other reasons. However, it is possible to construct the three-point functions explicitly, using the Coulomb gas representation [Vl. S. Dotsenko and V. A. Fateev, Nucl. Phys. B **240**, 312 (1984)]

to see that this does not happen.

[13]Negative values for x and c are, of course, not ruled out in a theory with nonpositive definite Boltzmann weights.

[14]This is a special case of a general result of Dotsenko and Fateev (see Ref. 12).

[15]Normalization is such that $\langle\phi(z_1)\phi(z_2)\rangle \sim |z_{12}|^{4/5}$.

[16]D. A. Kurze and M. E. Fisher, Phys. Rev. B **20**, 2785 (1979).

[17]J. L. Cardy, J. Phys. A **15**, L593 (1982); N. Breuer and H. K. Janssen, Z. Phys. B **48**, 347 (1982).

[18]D. Dhar, Phys. Rev. Lett. **51**, 853 (1983).

[19]R. J. Baxter, J. Phys. A **13**, L61 (1980).