Energy-Level Statistics of Integrable Quantum Systems

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Using a simple example we show that the distribution for the energy levels for integrable systems is not the uncorrelated Poisson distribution as is commonly believed. In particular, the spectrum was found to be rather rigid. We conjecture that these are typical properties of the integrable quantum systems.

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Energy-level statistics is an important property of many physical systems such as complex atoms and molecules, heavy nuclei, etc. This problem has recently attracted much attention[1–18] among physicists, chemists, and mathematicians. In particular, energy-level statistics provides an indication of the type of motion of a quantum system. It is commonly believed that "level repulsion," i.e., Wigner statistics for level spacings, is related to nonintegrable, chaotic classical motion, while the lack of repulsion, i.e., Poisson statistics, corresponds to integrable motion. Actually, this is no absolute rule; indeed the energy-level repulsion which appears in classically nonintegrable systems, as confirmed by several numerical computations, is due to some real interaction among the unperturbed states which leads to a formation of eigenstates which are the superposition of many unperturbed states. However, a repulsion which may be said to have a *kinematical* rather than *dynamical* origin may take place in integrable systems. (To avoid any possible misunderstanding we will use the concepts of integrability, etc., only in connection with the corresponding classical system.) The simplest example of such a kinematical repulsion is the one-degree-of-freedom conservative system. From the viewpoint of level statistics one could say that there is a strong repulsion in this case since the spacings are equal to the frequency of the classical motion which is typically different from zero. A more interesting example of kinematical repulsion has been given by Berry and Tabor[2] in a two-degrees-of-freedom harmonic oscillator.

The distinction between integrable and nonintegrable systems becomes much less clear when higher-order statistics, i.e., correlations between many levels, are taken into account. In the search for distinctive properties, spectral sequences of simple model systems have been subjected to various statistical tests. For example, Bohigas, Giannoni, and Schmit[16] were able to establish a definite similarity of fluctuation properties between the spectral sequence of Sinai's billiard and strings of eigenvalues of random matrices in the Gaussian orthogonal ensemble.

In this Letter we communicate the results obtained by statistically processing a string of 10^5 eigenvalues of the rectangular incommensurate billiard, obtained by reordering the double sequence

$$E_{mn} = \alpha m^2 + n^2, \qquad (1)$$

with α an irrational number.

In a previous paper,[11] we discussed the algorithmic properties of the same sequence in order to show that it is not a truly random one. Not being based on specific tests, our argument may leave the doubt that, nevertheless, the sequence may appear "random" to empirical tests. Thus, we have performed the following tests:

(1) *Distribution of the level spacings.*—This is shown in Fig. 1 and it looks fairly close to the Poisson distri-

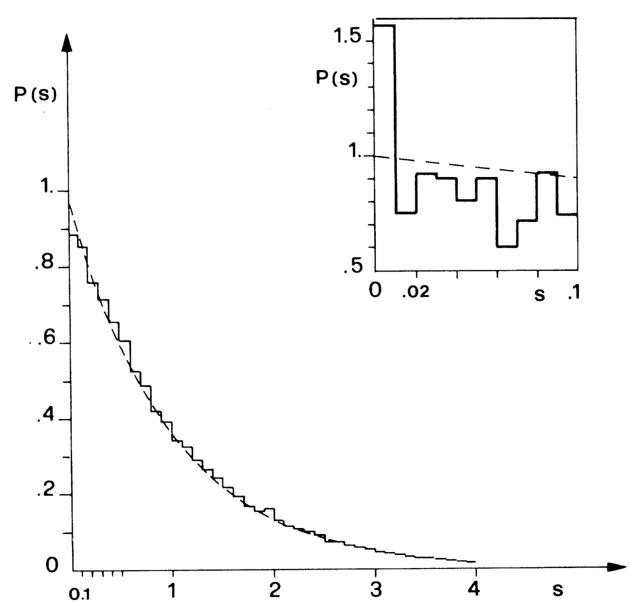

FIG. 1. Level-spacing distribution obtained from the first 100 000 levels (1) with $\alpha = \pi/3$. The dotted line is the Poisson distribution $P(s) = e^{-s}$.

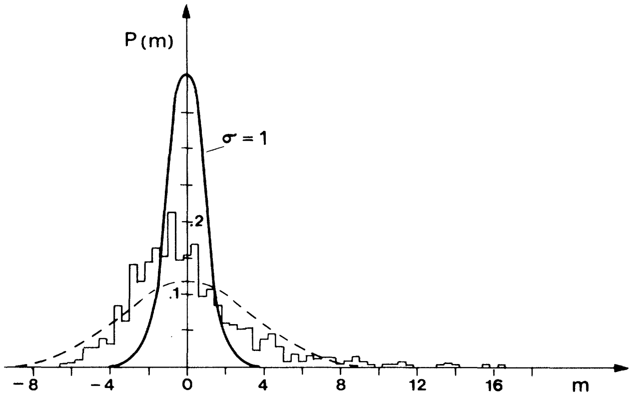

FIG. 2. Histogram of the distribution of deviations $m_i = (n_i^{ob} - n^{ex})/(n^{ex})^{1/2}$ of the observed number of spacings n_i^{ob} from the expected number n^{ex} in the ith interval. The intervals are so chosen that $n^{ex} = 90$ for each. The full line shows the Gaussian distribution of width $\sigma = 1$ corresponding to uncorrelated Poisson statistics; the actual observed rms width is $\sigma = \sqrt{11}$ (dashed line).

bution. Nevertheless, for small spacings we found statistically reliable deviations from Poisson's uncorrelated statistics. In fact, for the first interval in Fig. 1 the deviation of the number of spacings from the expected value is approximately 17 times the standard deviation.

The distribution inside this interval is also shown in Fig. 1. Again, the first interval of this distribution shows the largest fluctuation, with the actual number of spacings now larger than expected, approximately 18 times the standard deviation.

The χ^2 value for all ten subintervals is approximately 626 and even if we exclude the first subinterval it is still 297 which corresponds to a negligible confidence level. Apart from the whole interval (0,0.1) the agreement with Poisson's law seems to be rather good judging by Fig. 1. However, the calculated χ^2 value for 21 intervals is again too large, 69.8, corresponding to a confidence level of $\sim 10^{-7}$. This is another indication that the sequence is not completely random: In fact, it exhibits *too-large* fluctuations for a random sequence (a similar observation was made in Ref. 17). This is especially clear in the distribution of deviations from the Poisson law (Fig. 2). Not only are there substantial deviations from the Gaussian shape but what is more important, the width of the distribution is about 3.3 times larger than the expected one. This implies that the entire distribution is definitely different from uncorrelated Poisson statistics. In terms of the χ^2 test a value of $\sim 10\,000$ was obtained for 900 intervals which corresponds to a completely negligible confidence level.

(2) Δ_3 *statistics of Dyson and Mehta.*[19]—This characterizes the long-term correlations between levels, or the so-called "rigidity" of the spectrum. Specifically, for a given number L of levels we computed an average $\bar{\Delta}_3(L)$ in two different ways: (a) by averaging $\Delta_3(E_m,L)$ computed along a segment of L levels starting from level E_m, over a string $E_{n_1} \leq E_m \leq E_{n_2}$ ("spectral average"). In this case the value $\alpha = \pi/3$ was taken; (b) by averaging Δ_3 over a number of different values of α chosen at random in a given interval ("ensemble averaging").

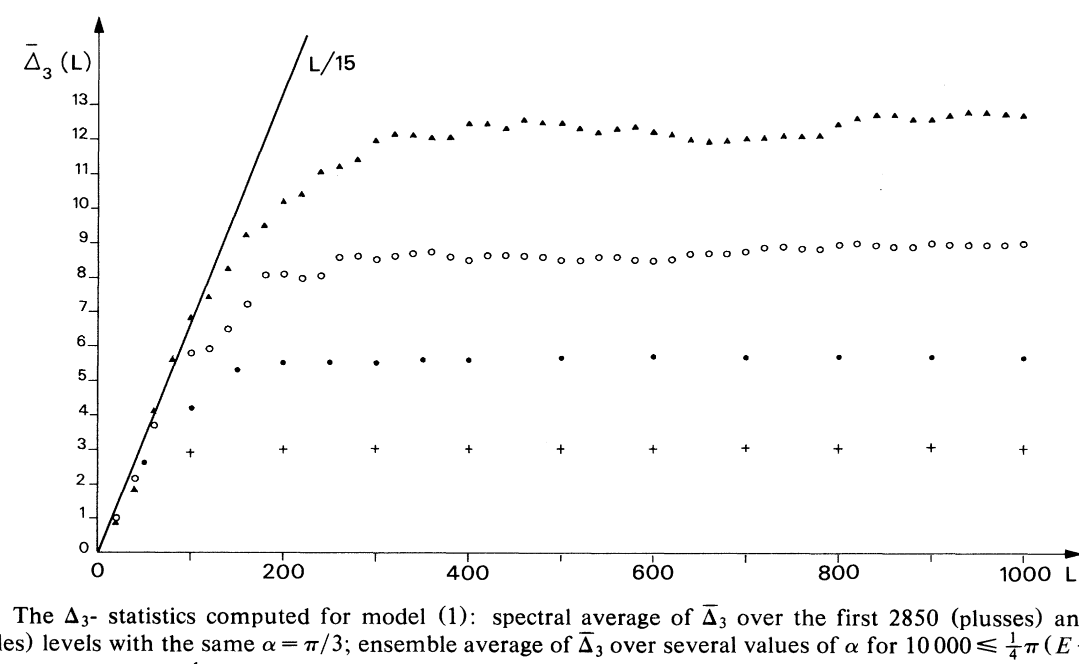

FIG. 3. The Δ_3- statistics computed for model (1): spectral average of $\bar{\Delta}_3$ over the first 2850 (plusses) and over 10 000 (solid circles) levels with the same $\alpha = \pi/3$; ensemble average of $\bar{\Delta}_3$ over several values of α for $10\,000 \leq \frac{1}{4}\pi(E+L) \leq 11\,000$ (open circles) and $20\,000 \leq \frac{1}{4}\pi(E+L) \leq 21\,000$ (triangles). The straight line shows $\bar{\Delta}_3 = L/15$.

The results so obtained for $\overline{\Delta}_3(L)$ have been plotted against L in Fig. 3. The straight line, $\overline{\Delta}_3 = L/15$, corresponds to the behavior of $\overline{\Delta}_3(L)$ for a Poisson statistics of the level spacings. For small L, $\overline{\Delta}_3(L)$ is close to this line, but then a kind of saturation occurs. Thereafter, $\overline{\Delta}_3(L)$ becomes a very slowly increasing function, such as one would expect of a rather regular sequence. On the other hand, if one looks at the set of eigenstates on the (n,m) plane which form a perfectly regular lattice (Fig. 4), one is led, indeed, to expect $\overline{\Delta}_3(L) \approx$ const or, at most, a very slowly increasing function.

A unified quantitative description of these results can be provided as follows: Consider a ring $E_1 \leqslant E \leqslant E_2$ inside which there are L levels, with $L \simeq \frac{1}{4}\pi(E_2 - E_1)$. Then consider a boundary layer of width ϵ along each of the two borders of the ring (Fig. 4). As a result of the irregular crossings of the curve $E =$ const by points of the lattice, the number of levels in the layer fluctuates. Actually, provided that ϵ is small ($\epsilon << 1$), we may assume that these levels come roughly as if at random. Hence $\Delta_3(E,L)$ computed over a string of L such levels, starting from level E, behaves like $L/15$. On the other hand, as a result of the regularity of the lattice, we can not expect the same for too-long strings; for long strings, $\overline{\Delta}_3(L)$ will be approximately $\frac{1}{15}$ of the "effective nonrigid length" of the string, corresponding to some effective critical value of $\epsilon = \epsilon_{cr}$ which can be determined by numerical experiments. The total number of these "random" levels lying in the two boundary layers near E_1 and E_2 is approximately $\frac{1}{2}\pi\epsilon_{cr}(\sqrt{E_1} + \sqrt{E_2})$ for $\alpha \approx 1$, so that we expect

$$\Delta_3(E,L) \approx \tfrac{1}{15} \times \tfrac{1}{2}\pi\epsilon_{cr}[\sqrt{E} + (E + 4L/\pi)^{1/2}] \quad \text{for } L \gtrsim \pi\epsilon_{cr}\sqrt{E}. \tag{2}$$

Instead, if $L \lesssim \pi\epsilon_{cr}\sqrt{E}$, we expect $\Delta_3(E,L) \approx L/15$.

By averaging these expressions of Δ_3 in the two ways (a) and (b) described above, we obtain analytical estimates to be compared with the numerical data of Figs. 3 and 5. In particular, from the spectral average of $\Delta_3(L)$ of Fig. 3, we obtain $\epsilon_{cr} \simeq 0.4$.

For an accurate check of the square-root dependence on L it is convenient to take $E = 0$ in expression (2), and compute the ensemble average, which gives

$$\overline{\Delta}_3(L) = (\epsilon\sqrt{\pi}/15)\sqrt{L}. \tag{3}$$

Figure 5 shows $\overline{\Delta}_3(L)$ averaged over twenty values of α within the interval (0.9,1.2). It is seen that the square-root dependence is verified with quite good accuracy. Moreover, fitting expression (3) to numerical results gives $\epsilon_{cr} = 0.5$, close to the value obtained from spectral averaging over different segments of levels with the same α.

In conclusion it appears that, at least for the integrable systems discussed here, the level sequence is rather rigid overall but behaves as a random one over small energy intervals. In particular this explains the irregular behavior of spacings leading roughly to the Poisson distribution.

The argument presented here for the two-dimensional case may be easily generalized to the N-

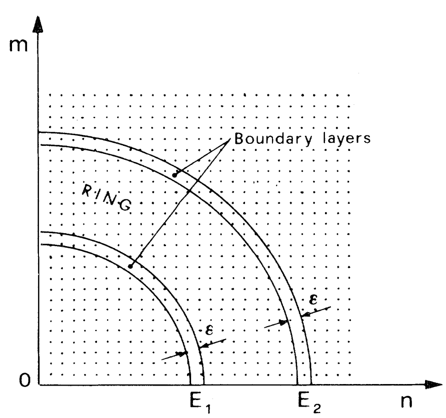

FIG. 4. The set of eigenstates.

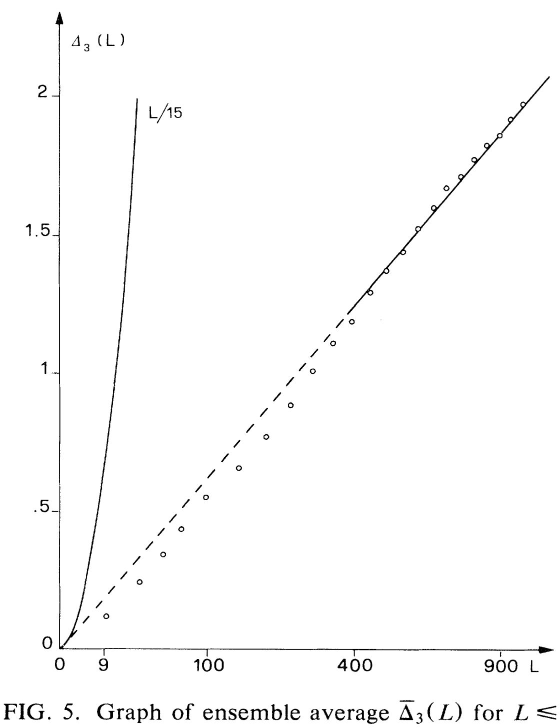

FIG. 5. Graph of ensemble average $\overline{\Delta}_3(L)$ for $L \leqslant 1000$ showing the square-root dependence on L; $\alpha \approx 1$. The straight line fits the numerical data with $\epsilon_{cr} = 0.53$.

dimensional case and gives

$$\bar{\Delta}_3(L) \approx \begin{cases} (\sqrt{N}/15)L^{(N-1)/N} & \text{for } L \gtrsim N^{N/2}, \quad N >> 1, \\ L/15 & \text{for } L \lesssim N^{N/2}, \end{cases} \tag{4}$$

assuming $E=\sum_{i=1}^{N}\alpha_i\eta_i^2$, $\alpha_i \approx 1$, and ϵ_{cr} independent of N. Even though the expressions derived for the level statistics are related to the particular type of spectrum, (1), we conjecture that the qualitative structure of the spectrum would be the same for a typical integrable many-dimensional system. This view is supported, particularly, by the results of Ref. 17 where a similar behavior has been observed in a different model.

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