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Fractional Statistics of the Vortex in Two-Dimensional Superfluids

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The quantum behavior of two identical point vortices (e.g., in a superfluid ^{4}He thin film) is studied. It is argued that this system obeys neither Bose nor Fermi statistics, but intermediate or θ statistics: We find that a single vortex in this system possesses *quarter-fractional* statistics (i.e., $\theta=\pi/2$ or $3\pi/2$). The source of the θ statistics is identified in the relative zero-point motion of the vortices.

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Topology has a profound effect on quantum mechanics, as demonstrated, for example, by the Aharonov-Bohm effect. When one pierces a hole in a two-dimensional manifold, the topology changes from a trivial singly connected one to a nontrivial multiply connected one. A quantum-mechanical vortex has effectively such a hole at its center. Consequently, a quantum system of N identical vortices in two dimensions, such as those in a thin film of superfluid helium, possesses a multiply connected configuration space, M_N.[1,2] The fundamental group of M_N, $\pi_1(M_N)$, consisting of the different homotopy classes of paths in M_N, is isomorphic to the infinite non-Abelian braid group, $B_N(R^2)$, leading to the possibility of θ statistics.[1-3] Studies of certain $(2+1)$-dimensional models, such as flux-tube–charge composites[4,5] and skyrmions,[5,6] also lead to these exotic statistics, which interpolate between bosons and fermions. The $\frac{1}{3}e$-charged excitations in the quantum Hall fluid[7,8] and the 't Hooft–Polyakov monopole[9] have also been proposed as candidates for θ statistics.

Here we explore the possibility that a vortex in two-dimensional quantum systems, such as superfluid helium thin films, obeys θ statistics. Such a vortex is a topologically nontrivial soliton in two dimensions, and hence is a good candidate for these statistics. We quantize the classical motion of two identical point vortices in an unbounded thin film, and find that an individual vortex in this system possesses quarter-fractional statistics. These results may lead to experiments on θ statistics.

The two-dimensional classical Hamiltonian system of N massless point vortices of circulation κ_i centered at (x_i,y_i) $i=1,2,\ldots,N$, in an unbounded flat, frictionless, incompressible, fluid thin film of thickness δ and density ρ obeys the equations of motion[10]

$$\kappa_i\, dx_i/dt=+\partial H_*/\partial y_i, \tag{1}$$

$$\kappa_i\, dy_i/dt=-\partial H_*/\partial x_i, \tag{2}$$

where, after subtraction of the self-energy of individual vortices, and definition of a as a scale factor,

$$H_*=-(1/4\pi)\sum_{i<j}\kappa_i\kappa_j\ln(r_{ij}^2/a^2), \tag{3}$$

$$r_{ij}^2=(x_i-x_j)^2+(y_i-y_j)^2. \tag{4}$$

The energy of the system is given by $H=(\rho\delta)H_*$. In the case of superfluid helium, all κ_i are quantized with values $\pm h/m$, where h is Planck's constant and m is the mass of the helium atom. The validity of the Hamiltonian, Eq. (3), is borne out by the success of the Kosterlitz-Thouless theory[11] in predicting the two-dimensional phase transition in thin films of superfluid helium.[12] We expect that this hydrodynamic Hamiltonian will remain valid as long as the intervortex distance r_{ij} is greater than the diameter of the vortex core; solitons whose separations are large compared with their core sizes are approximated as point particles. Also, the thickness of the thin film is assumed to be about a monolayer. Thus, motions perpendicular to the substrate can be ignored. The resulting motion of the system is two dimensional. We assume that the temperature is arbitrarily close to absolute zero.

The dynamical system of two identical vortices is described by the four equations of motion, Eqs. (1) and (2) with $i=1$ and 2. Let us introduce the relative coordinates $x=x_1-x_2$ and $y=y_1-y_2$, and the center-of-vorticity coordinates $X=(x_1+x_2)/2$ and $Y=(y_1+y_2)/2$, of the two vortices. Then Eqs. (1) and (2) become

$$\kappa\, dx/dt=+2\,\partial H_*/\partial y, \tag{5}$$

$$\kappa\, dy/dt=-2\,\partial H_*/\partial x, \tag{6}$$

where $\kappa=h/m$ and where

$$H_*=-(\kappa^2/4\pi)\ln[(x^2+y^2)/a^2], \tag{7}$$

while X and Y become constants of motion.

In order to examine the angular momentum properties of this vortex system, let us transform this classical Hamiltonian system to polar coordinates $r=(x^2+y^2)^{1/2}$, the radial separation, and $\phi=\tan^{-1}(y/x)$, the orientation angle of the two vortices. Using the chain rule, one transforms Eqs. (5) and (6) to

$$\kappa\, dr^2/dt=+4\,\partial H_*/\partial\phi, \tag{8}$$

$$\kappa\, d\phi/dt=-4\,\partial H_*/\partial r^2. \tag{9}$$

The transformation is canonical. Thus, ϕ and $-(\kappa\rho\delta)r^2/4$ can formally be viewed as the canonical coordinate and momentum, respectively, in Hamilton's equations. Also, because of the rotational symmetry of the system, $\partial H_*/\partial\phi=0$, so that from Eq. (8) we see that r^2 is a constant of the motion. Hence, classically, the two vortices orbit around each other in a circle of diameter r.[13] The physical significance of r^2 is that it is proportional to the angular momentum of the two-vortex system. It should be emphasized that there actually exists angular momentum *in the superfluid* due to the orbiting of the two vortices, which is given by[14]

$$L_z=-(\kappa\rho\delta)r^2/4. \tag{10}$$

Since the superfluid is composed of atoms whose angular momentum is quantized, and all the atoms are in a single coherent state, we argue that the angular momentum L_z in the moving superfluid must also be quantized. Furthermore, quantization of angular momentum in superfluid ^{4}He has been seen experimentally.[15] Thus, we use the commutator[16] $[\phi,L_z]=i\hbar$, which leads to

$$[r^2,\phi]=iC', \tag{11}$$

where $C'=4\hbar/\kappa\rho\delta$, a result which also follows from canonical quantization of Hamilton's equations (8) and (9). The simplest ϕ representation of the r^2 operator is

$$r^2=C'i\,d/d\phi=-(C'/\hbar)L_z, \tag{12}$$

where L_z now is the *local* generator of rotation.[2] The eigenvalues of r^2 are given by

$$r^2\Psi_M(\phi)=C'M\Psi_M(\phi), \tag{13}$$

where the eigenfunctions are

$$\Psi_M(\phi)\propto\exp(-iM\phi), \tag{14}$$

and M is the angular momentum quantum number. However, M need not be an integer since $\Psi_M(\phi)$ need not in general be single valued, when θ statistics holds. Instead, we demand that[1,2,4,6]

$$\Psi_M(\phi+\pi)=e^{-i\theta}\Psi_M(\phi) \tag{15}$$

since a rotation of the system through 180° interchanges the positions of the two identical vortices. Thus,

$$e^{-iM\pi}=e^{-i\theta}e^{-i2\pi n'},\quad n'\text{ integer}, \tag{16}$$

implying that

$$M=2n'+\theta/\pi, \tag{17}$$

where θ is as yet undetermined.

However, there exists an ambiguity in θ in the polar coordinate system as a result of the well-known mathematical ambiguity in ϕ at the origin $(r,\phi)=(0,\phi)$ of these coordinates. The physical meaning of this mathematical ambiguity is that the orientation of the core of a single, coalesced vortex formed from the merging together of the two vortices is undefined. But to an observer at infiinity, the two vortices appear as if they were a single, coalesced vortex at the origin. Hence, this ambiguity is equivalent to an ambiguity in the orientation of the coordinate system used by the observer at infinity. Quantum mechanically, such an ambiguity shows up as an arbitrary phase factor. Hence, θ is ambiguous. However, no such orientational ambiguity arises in the rectangular coordinate system, since the origin $(x,y)=(0,0)$ of these coordinates is now well defined. Hence, in order to eliminate this ambiguity, we transform from polar to Cartesian coordinates at the quantum mechanical level using

$$x = (e^{i\phi/2} r e^{i\phi/2} + e^{-i\phi/2} r e^{-i\phi/2})/2, \qquad (18)$$

$$y = (e^{i\phi/2} r e^{i\phi/2} - e^{-i\phi/2} r e^{-i\phi/2})/2i, \qquad (19)$$

where x, y, r, ϕ are all operators. The specific choice of order of operators in Eqs. (18) and (19), using Eq. (12), gives

$$x^2 + y^2 + 2\lambda [x,y]/i = r^2 \qquad (20)$$

with the parameter $\lambda = 0$. Other orderings of the operators can give $\lambda \neq 0$, but then Pythagoras's theorem breaks down at small distances, a possibility that can be ruled out for *pointlike* vortices, because of the assumed flatness of the thin film. (However, we leave open the possibility that $\lambda \neq 0$ for vortices with nontrivial internal structure of their cores.) Also, using Eqs. (12), (18), and (19), we deduce

$$[x,y] = iC'', \qquad (21)$$

where $2C'' = C'$, a result which also follows from canonical quantization of Hamilton's equations (5) and (6).[17] The simplest x representation of the operator y is

$$y = (C''/i)\, d/dx, \qquad (22)$$

which operates on the wave function $\Psi(x)$.[18] Then the operator $x^2 + y^2$ becomes formally identical to the Hamiltonian of a one-dimensional simple harmonic oscillator. Hence, the solution to the angular momentum eigenvalue problem is

$$(x^2 + y^2)\Psi_n(x) = 2C''(n + \tfrac{1}{2})\Psi_n(x), \qquad (23)$$

where

$$\Psi_n(x) \propto H_n(\alpha x) \exp(-\tfrac{1}{2}\alpha^2 x^2), \qquad (24)$$

with $\alpha = (\pi\rho\delta/m)^{1/2}$ and $H_n(\alpha x)$ the Hermite polynomials.[19] In order that $|\Psi_n(x)|^2$ remain bounded as $x \to \pm\infty$, it should be emphasized that n must be an exact integer. Furthermore, the fraction $\frac{1}{2}$ in $(n + \frac{1}{2})$ is a necessary conseqence of the commutator, Eq. (21), and represents the relative zero-point motion of the two vortices. Comparing Eqs. (13) and (23), we obtain

$$M = n + \tfrac{1}{2} = 2n' + \theta/\pi, \qquad (25)$$

so that either

$$n \text{ is even, and } \theta = \pi/2, \qquad (26)$$

or

$$n \text{ is odd, and } \theta = 3\pi/2. \qquad (27)$$

Thus, in either case, alternate angular momentum states are missing as a result of statistics. The theory so far does not determine which θ is the actual one. In either case, a single vortex in this system is predicted to be neither a boson nor a fermion; rather, it should obey *quarter-fractional* statistics.

The center of vorticity will also undergo quantized motion. To see this, let us place the two-vortex system near the center of a large circular, *nonrotating* hoop of radius R_0. Classically, the center of vorticity at (X,Y), or (R,Φ), where $R^2 = X^2 + Y^2$ and $\Phi = \tan^{-1}(Y/X)$, behaves like a single vortex of circulation 2κ which orbits in a circle of radius R around the center of the hoop with angular momentum[14] $-(\kappa\rho\delta)R^2$. Again, one quantizes this angular momentum through the commutators

$$[\tfrac{1}{2}R^2, \Phi] = [X,Y] = iC, \qquad (28)$$

where $4C = C''$. The resulting angular momentum of the center of vorticity, and hence of a single, isolated vortex, is *half-fractional.* The total angular momentum of the system is *integral,* and thus the *total* wave function is single valued.[20]

Recently, we have obtained the result $\theta(N) = \pi/N$, for a system of N point vortices. This is analogous to the result of Ref. 9 for N 't Hooft–Polyakov monopoles. The special case $N = 1$ agrees with the fermionic nature of a solitary Abrikosov vortex,[21] and the above result for the single, isolated vortex. The special case $N = 2$ is consistent with the results of this paper.

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[1] J. M. Leinaas and J. Myrheim, Nuovo Cimento B **37**, 1 (1977); G. A. Goldin, R. Menikoff, and D. H. Sharp, J. Math. Phys. **22**, 1664 (1981); Y. S. Wu, Phys. Rev. Lett. **52**, 2103 (1984).

[2] G. A. Goldin and D. H. Sharp, Phys. Rev. D **28**, 830 (1983).

[3] That is, under the interchange of two identical particles at positions 1 and 2, $\Psi(2,1) = \exp(-i\theta)\Psi(1,2)$, where $0 \leq \theta < 2\pi$, for the sense of interchange given by a classically allowed path.

[4] F. Wilczek, Phys. Rev. Lett. **49**, 957 (1982).

[5] Y. S. Wu, Phys. Rev. Lett. **53**, 111 (1984).

[6] F. Wilczek and A. Zee, Phys. Rev. Lett. **51**, 2250 (1983).

[7] B. Halperin, Phys. Rev. Lett. **52**, 1583 (1984).

[8] D. Arovas, J. R. Schrieffer, and F. Wilczek, Phys. Rev.

Lett. **53**, 722 (1984).
[9]G. A. Ringwood and L. M. Woodward, Phys. Rev. Lett. **53**, 1980 (1984).
[10]L. Onsager, Nuovo Cimento Suppl. **6**, 279 (1949).
[11]D. R. Nelson and J. M. Kosterlitz, Phys. Rev. Lett. **39**, 1201 (1977).
[12]V. Kotsubo and G. A. Williams, Phys. Rev. Lett. **53**, 691 (1984).
[13]G. K. Batchelor, *An Introduction to Fluid Dynamics* (Cambridge Univ. Press, Cambridge, 1967), p. 531. This motion is equivalent to that of two *massless* line charges of like sign orbiting around each other in a uniform magnetic field directed along their common axis.
[14]H. Lamb, *Hydrodynamics* (Dover, New York, 1945), 6th ed., p. 215. The angular momentum is given relative to that of a vortex of circulation 2κ at the center of vorticity.
[15]G. B. Hess and W. M. Fairbank, Phys. Rev. Lett. **19**, 216 (1967).
[16]P. Carruthers and M. M. Nieto, Rev. Mod. Phys. **40**, 411 (1968).
[17]Here Eq. (21) is a result of the quantization of the relative angular momentum of the two-vortex system. It has been shown to follow in another way for He II vortices from Dirac's method of constrained Hamiltonian systems by M. Rasetti and T. Regge, Physica (Utrecht) 80A, 217 (1975), and applied by A. L. Fetter, Phys. Rev. **162**, 143 (1967), and **163**, 391 (1967), and **186**, 128 (1969); G. E. Volovik, Pis'ma Zh. Eksp. Teor. Fiz. **15**, 116 (1972) [JETP Lett. **15**, 81 (1972)]; and L. Mittag, M. J. Stephen, and W. Yourgrau, in *Variational Principles in Dynamics and Quantum Theory,* edited by W. Yourgrau and S. Mandelstam (Dover, New York, 1979), 3rd ed., p. 161.
[18]The simultaneous operations $x \rightarrow -x$ and $y \rightarrow -y$ required for interchange of the two vortices cannot be simultaneously applied, since $\Psi(x)$ is a function of x alone, and not of x and y. Since $[x,y]\neq 0$, knowing x means that y is completely uncertain. Hence, the concept of singlevaluedness of $\Psi(x)$ over the x-y plane is ill defined. Also, the x parity of $\Psi(x)$ is *not* the quantum number of interchange. However, in polar coordinates, the *single* operation $\phi \rightarrow \phi + \pi$ specifies interchange, and the quantum number of interchange is well defined.
[19]The Gaussian wave-packet size $2/\alpha$ is about 56 Å for the thinnest of the superfluid ^{4}He films studied by B. C. Crooker, B. Hebral, E. N. Smith, Y. Takano, and J. D. Reppy, Phys. Rev. Lett. **51**, 666 (1983). This is much larger than the vortex-core size, thus justifying the point-vortex approximation.
[20]One can also fix θ by demanding that the product wave function for the relative and center-of-vorticity motions in polar coordinates be single valued. Then, it is not necessary to transform into rectangular coordinates to fix θ.
[21]R. Jackiw and A. N. Redlich, Phys. Rev. Lett. **50**, 555 (1983).