Low-Frequency Relaxation in Ising Spin-Glasses

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For d-dimensional short-range Ising spin-glasses with local spin-flip dynamics, the correlation function $\langle s_i(0)s_i(t)\rangle$ is argued to be bounded below by a function of the form $\exp[-c(\ln t)^{d/(d-1)}]$ in a temperature range *above* the spin-glass transition. The slow relaxation of large isolated clusters of unfrustrated spins is responsible for this bound. We suggest that this is the signature for an intermediate Griffiths phase between the spin-glass and the paramagnetic phases.

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Spin-glasses are thought to undergo a phase transition in sufficiently many dimensions from a high-temperature paramagnetic phase with exponential relaxation into a low-temperature glassy phase with long-range order in time. Much recent progress has been made towards understanding the statics and dynamics in the low-temperature phase, especially in the mean-field limit.[1-4] The temperature range above the spin-glass transition has received much less theoretical attention.[5]

In this Letter, we demonstrate that the low-temperature phase necessarily melts into a phase whose correlation functions have nonexponential tails. In particular, we derive a lower bound for the correlation function

$$q(t)=\frac{1}{N}\sum_{i=1}^{N}\langle s_i(0)s_i(t)\rangle_T$$

in a short-range Ising spin-glass with local spin-flip dynamics. In d dimensions, we find[6]

$$q(t)\geq A\exp[-c(\ln t)^{d/(d-1)}], \tag{1}$$

where A and c are nonuniversal constants with $A>0$ for a range of temperatures above the glass transition. Note that the bound in (1) dies away more slowly than an exponential but faster than a power law. In particular, the currently popular "stretched" exponential[7,8] or Kohlrausch form $\exp[-(t/\tau)^{\beta}]$ is ruled out at long times for any $\beta>0$. We will show that the bound disappears to all orders in perturbations about high temperature and high dimension (which perhaps explains why it has escaped notice). This nonexponential relaxation is due to the same unfrustrated clusters which are responsible for the Griffiths singularity[9] in the magnetic susceptibility. We suggest that it is the dynamical signature of a "Griffiths phase," lying between the spin-glass and paramagnetic phases.

The physical ideal behind the bound in (1) is simple. Rare compact clusters of spins whose interactions are unfrustrated[10] lock into one of two configurations, and flip from one to the other very infrequently. These slow clusters give rise to nonexponential relaxation for all temperatures between the spin-glass transition temperature T_{SG} and the ferromagnetic transition temperature T_F for the pure system (i.e., for the unfrustrated regions of largest bond strengths). A result identical to (1) has been recently obtained for dilute ferromagnets by Dhar.[11,12] Langer and Kotliar[13] have been pursing path-integral saddle-point methods to calculate the asymptotic form of the decay of $q(t)$.

Consider the model defined by the Hamiltonian $H=\sum_{\langle ij\rangle}J_{ij}s_is_j$ and single spin-flip dynamics, where[14] $s_i=\pm 1$ and the summation extends over all pairs of nearest neighbors on a d-dimensional hypercubical lattice. The bond strengths J_{ij} are quenched, independent random variables.

We break up the sum over sites in the definition of $q(t)$ into two parts: one that includes all the compact unfrustrated clusters and the other over the rest of the system. Since the dynamics is relaxational, we expect (but have not proven) that for any given site i, the thermal average $\langle s_i(0)s_i(t)\rangle_T$ is positive for all $t\geq 0$. Thus the contribution of the unfrustrated clusters alone should provide a lower bound on the total relaxation $q(t)$. Let $P(L,J)$ be the probability that a given spin belongs to an unfrustrated cluster of L sites to a side, and characteristic bond strength J. This cluster will have two ground states (related by $s\rightarrow -s$ for all spins in the cluster). At finite temperatures and short times, fluctuations about these ground states are characterized by the magnetization $M(J/T)$ of the corresponding ferromagnet[15] with bond strength J. There is a free-energy barrier $F(L,J)$ between states of magnetization $\pm M$ and zero. The relaxation rate of the cluster, measured in units of the microscopic time scale for single spin flips, is given by the Arrhenius form $\exp[-F(L,J)/T]$, where T is the temperature (setting Boltzmann's constant equal to 1). Since these

clusters are very far apart, they relax independently, giving rise to the lower bound

$$q(t) \geq \int dL \int dJ\, M^2(J/T) P(L,J) \exp[-te^{-F(L,J)/T}]. \quad (2)$$

We shall first make a crude estimate for this bound, using a Gaussian distribution of bond strengths with zero mean and variance J_0^2 and considering only ferromagnetic clusters. Later we shall discuss other unfrustrated regions and possible boundary effects; these change only the constants A and c in (1). The probability of finding a spin in a cluster with bonds of strength $+J$ to within $\Delta J \ll J_0$ is given by

$$P(L,J) = L^d \left(\frac{\Delta J}{(2\pi)^{1/2} J_0} \right)^{dL^d} \exp\left(- \frac{dL^d J^2}{2J_0^2} \right). \quad (3)$$

[The number of bonds in the cluster is $d(L-1)L^{d-1} \approx dL^d$.] To flip the entire cluster over, we must sweep a domain wall across it. The barrier to be overcome is the surface free energy of a domain wall spanning the cluster:

$$F(L,J) \approx \sigma(J/T) L^{d-1}, \quad (4)$$

where σ is the surface tension.[16]

For $t \to \infty$, the integral in (2) may be evaluated by the method of steepest descents. One can immediately write down the two equations for the saddle point $L = L^*(t)$ and $J = J^*(t)$. It is convenient to first eliminate the time t between these two equations to obtain an equation for J^* as a function of L^*. Using (3) and (4), and neglecting the prefactor L^d in (3) which only gives rise to logarithmic corrections, we find that J^* is given by maximizing $[\sigma(J/T)]^{d/d-1}/[B+J^2]$ where $B = 2J_0^2 \ln[(2\pi)^{1/2} J_0/\Delta J]$. With use of standard properties of the surface tension[17,18] it is easy to see that one obtains a finite $J^*(T)$ (in any dimension $d > 20$)[19] which turns out to be independent of L. [Including subleading terms in L in (3) or (4) would make J^* weakly dependent on L.] The saddle-point equations may now be used to obtain $L^* \simeq [\ln t/\Sigma^*(T)]^{1/(d-1)}$, where $\Sigma^*(T) = \sigma(J^*(T)/T)/T$. Putting all this together yields an estimate of the form (1) with

$$c = (d/2J_0^2)\{B + [J^*(T)]^2\}[\Sigma^*(T)]^{-d/(d-1)}.$$

The preceding analysis is valid for arbitrary dimensions and is directly applicable to the mean-field limit $d \to \infty$. The optimal bond strength $J^*(T)$ and the corresponding reduced surface tension $\Sigma^*(T)$ are weak functions of dimension and approach finite limiting forms as d gets large. We thus find that $c \approx dK_1(T)$ as $d \to \infty$ in (1) and our estimate vanishes exponentially as d becomes large. Thus the nonexponential tail in the relaxation due to large unfrustrated clusters appears to be absent in mean-field theory[20] and the relaxation may well be exponential above the spin-glass transition. This would be in agreement with mean-field results both from functional-integral methods[1] and from linearized Glauber dynamics.[21,22]

Since the Gaussian distribution is unbounded there are clusters which are below their ferromagnetic transition temperature no matter how high the temperature. It can be shown as $T \to \infty$ that the optimal bond strength $J^*(T)$ increases linearly with the temperature (for $d > 2$) and we obtain (1) with $c \approx T^{2+d/(d-1)} \times K_2(d)$. Thus our bound vanishes also at high temperatures. It is worth emphasizing that in both the high-dimensionality (mean-field) and the high-temperature limits our estimate has an essential singularity. The nonexponential relaxation above the spin-glass transition thus appears to vanish order by order in perturbation theory in $1/d$ and $1/T$.

Let us now be more careful in the derivation of the bound in (1). There are two important issues that have not been addressed. First, using gauge invariance,[15] one can show that for every ferromagnetic cluster with L^d spins there are 2^{L^d-1} clusters of the same size which have the same activation energy for flipping between two low-energy configurations.[23] This roughly multiplies $P(L)$ by 2^{L^d}, subtracts $(\ln 2)/d$ from B, and thus changes only the constant c in (1). Secondly, we have ignored possible effects of the bonds connecting the cluster to the rest of the system. One could imagine that under some circumstances the environment of a cluster might conspire to "push" it over. This becomes particularly disturbing in high dimensions, where clusters develop very large surface areas. We can control this problem for Gaussian and flat bond distributions by surrounding the cluster with weak bonds. (We expect that the decays in the $\pm J$ model are the same, although we cannot control the effects of the boundaries so simply.) Surrounding the cluster with bonds of strength at most ΔJ multiplies B by $1+2/L$ in $P(L)$ and lowers the guaranteed free-energy barrier by $\Delta J\, dL^{-d-1}$, effectively lowering σ by $d\Delta J$. Again, the form (1) of our bound is unchanged (although the large dimension and temperature limits are modified somewhat).

For a bounded distribution of bonds, nonexponential relaxation of the form (1) is expected only in a finite range of temperatures (Fig. 1). Below the spin-glass transition temperature T_{SG}, $q(t)$ attains a nonzero value at large times and our bound becomes irrelevant. Above the ferromagnetic transition temperature T_F for the pure system with unfrustrated bonds of the maximum strength, the surface tension in all clusters vanishes and our lower bound becomes trivially zero. In this high-temperature paramagnetic phase the relaxation is presumably exponential in time. Between these temperatures, our analysis demonstrates the existence of an intermediate phase of nonexponentially relaxing correlations in time (Fig. 1).

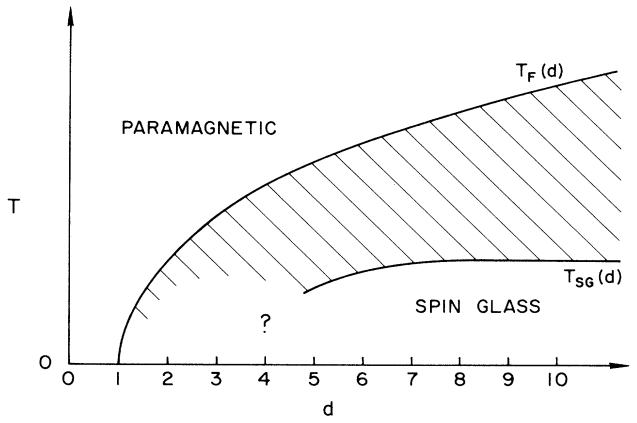

FIG. 1. Dimension-temperature phase diagram for a short-range Ising spin-glass with a bounded distribution of bond strengths. The intermediate phase indicated by hatched lines lies between the spin-glass transition temperature T_{SG} and the ferromagnetic transition temperature $T_F \approx J_{max}$.

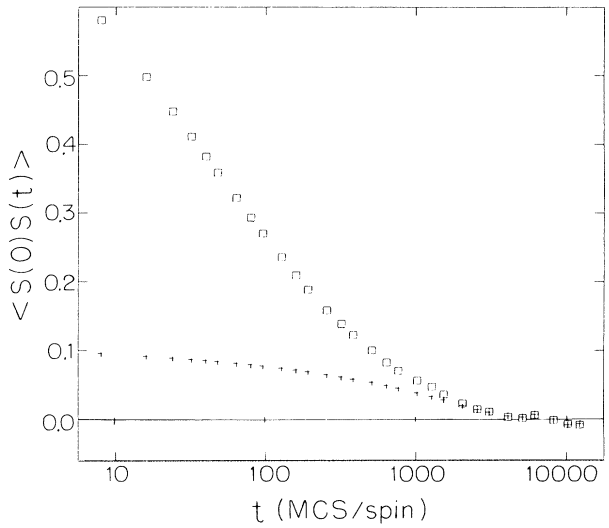

FIG. 2. Squares show the correlation function $q(t)$ vs time (in Monte Carlo steps per spin) for a two-dimensional, $\pm J$, equal-probability spin-glass on a 50×50 square lattice with periodic boundary conditions at $T = 1.2J$. ($T_F = 2.27J$ in this model, and T_{SG} is thought to be zero in two dimensions.) Crosses show the contribution to $q(t)$ from 311 spins immediately surrounding the four slowest unfrustrated clusters on the lattice. Averages were made over all appropriate initial times on four separate runs on the same lattice; the runs totaled over 2^{16} Monte Carlo steps per spin.

It is generally acknowledged that the free energy in this temperature region has an essential singularity (the Griffiths singularity) as a function of the external magnetic field,[9] and an essential singularity at T_F even at zero field.[24] Since the hexatic phase in two-dimensional melting also possesses only an essential singularity in the free energy,[25] we see no excuse for not calling this intermediate-temperature region a phase. The analogy can be pushed further. The glass transition is associated with order in time; two-dimensional melting is associated (at least partly) with orientational order in space. Both the hexatic and the Griffiths phases lie between a high-temperature phase with no long-range order and a low-temperature phase with long-range order. And, as we have just argued for spin-glasses, both possess order which decays to zero in a nonexponential fashion.

From an experimental point of view, the main implication of our results is that exponential relaxation just above the spin-glass transition is ruled out. It is not certain that large compact clusters dominate the relaxation at very long times, and thus it is possible that the relaxation will be slower than the bound (1), perhaps algebraic. [In physical materials inhomogeneities[5] in the sample would be expected to enhance the importance of clusters, and swamp our bound (1).] Whether the magnitude of the effect will be visible is of course a separate issue. On the one hand, Griffith singularities are notoriously difficult to see in experiments. The largest clusters to flip, in samples of macroscopic dimensions and after macroscopic times, will still be quite small (optimistically containing no more than 10^2 spins). The predicted singularity in the susceptibility has never been observed.[26] On the other hand, the slow relaxation of the unfrustrated compact clusters is dramatically visible in $d = 2$, $\pm J$ Monte Carlo simulations.[27] Figure 2 shows $q(t)$ for a two-dimensional, $\pm J$ model on a 50×50 square lattice with periodic boundary conditions, for $T = 1.2J$. Also shown in the contribution of 311 spins (12.4% of the total) which form the immediate environments of four unfrustrated clusters. (Spins within a correlation length or so[28] of a slow cluster will also show slow relaxation.) For $t > 1000$ Monte Carlo steps per spin, the dominant contribution to the correlation function comes from these four regions. Since the exponentially rare unfrustrated clusters contribute to the correlation function $q(t)$ for an exponentially long time before they flip, observing this phase by measuring dynamical correlations should be much easier than by measuring static susceptibilities. What this intermediate phase does to the spin-glass melting transition is open to conjecture.

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[14]It is possible to obtain (1) for the case of soft spins; i.e., the restriction $s_i = \pm 1$ can be replaced by suitable quadratic and quartic terms in the Hamiltonian. The only effect of this change, from our point of view, is to modify the surface tension. This modification does not affect the qualitative properties of the surface tension that we will assume for our analysis; it will only change the constant c in (1).

[15]The spin-glass Hamiltonian is invariant under a local gauge transformation which for a fixed spin **i** flips that spin and the signs of all the bonds connected to it: $s_i \rightarrow -s_i$ and $J_{ij} \rightarrow -J_{ij}$ for all j. An unfrustrated cluster is equivalent to a ferromagnetic one via a gauge transformation. See Ref. 8.

[16]More precisely, $F(L)=\sigma(J/T,L)L^{d-1}$, where σ is reduced slightly because of reduced ordering at the boundaries, but approaches the usual surface tension in the thermodynamic limit $L \rightarrow \infty$.

[17]The surface tension $\sigma(J/T)/J$ measured in units of the coupling J for a d-dimensional Ising model is expected to be a continuous, monotonically decreasing function of the temperature T/J, which approaches a finite value as $T/J \rightarrow 0$ and is identically zero above the ferromagnetic transition temperature $T_F \approx J$.

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[19]In two dimensions the optimal J^* goes to infinity for a Gaussian bond distribution. If the distribution is narrower than Gaussian (e.g., bounded), our analysis holds in two dimensions. Conversely, if it is sufficiently broad (e.g., Lorentzian) then even in higher dimensions the long-time tail will be significantly longer than indicated by (1). In any case, bonds of an arbitrary fixed strength may be used to get a lower bound on $q(t)$ though it will not be sharp.

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