Effective-Action Expansion in Perturbation Theory

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(Received 14 December 1984)

I present a systematic method for the development of the effective-action expansion in perturbation theory. The multiderivative terms beyond the effective potential can be evaluated in a direct and simple manner relying only on the familiar momentum space and Feynman propagator. I have used the self-coupled scalar field to explain the details of this new formulation. The effective-action expansion for this model in the one-loop approximation is evaluated up to the terms containing four derivatives. This method can be readily generalized to other models with spins and internal symmetries.

PACS numbers: 11.10.Ef, 03.65.Db, 03.70.+k, 11.10.Lm

It has always been an important problem in physics to extract the dominant contribution of the short-distance effects on the large-distance behavior. A classical example is the multipole expansion. In the local relativistic quantum field theory the short-distance effects are mostly due to quantum fluctuations. In the low-energy limit, short-distance effects are not explored in detail. Typical examples are the effects of heavy particles, confining particles, and quantum fluctuation of the light particles. It is more convenient to eliminate those degrees of freedom not directly observable and incorporate their effects into an effective action of the observable fields. The expansion of this necessarily nonlocal effective action into an infinite series of local actions in the order of the number of space-time derivatives is known as the effective-action expansion.

This effective-action expansion is best formulated in the functional integral method in which the unobserved fields are integrated out. The calculation of the effective potential, which is the leading term of the expansion with no derivative, is well known.[1] However, the multiderivative terms in the expansion have been investigated only in a very limited way.[2–4] There is no simple, systematic method to develop the series of the effective-action expansion. In the standard calculation, it requires that either the fields or the derivatives of the fields are constant. This procedure would eventually lead to unacceptable consequences technically and conceptually. Ilipoulos, Itzykson, and Martin introduced an alternative method and succeeded in obtaining the next term of the expansion containing two derivatives in the ϕ^4 theory.[5] However, their methods are too complicated to be generalized systematically for higher-order terms of the expansion. Recently phenomenological Lagrangians containing four derivatives, such as the Skyrme model and the Wess-Zumino term, have been instrumental in opening up a new direction of baryon physics.[6] It is possible that these extra four-derivative terms can be obtained from an effective-action expansion.[3,7,8,9] To answer this and other interesting questions it becomes necessary to develop a better treatment for the effective-action expansion.

In this paper, I shall present a systematic method for the effective-action expansion which can be applied to any given theory. Since the most crucial development of the method can be appreciated without the unnecessary complication of spins and internal symmetries, I shall use the self-coupled neutral scalar theory in D dimensions to facilitate the presentation. The effective-action expansion up to the terms containing four derivatives will be evaluated in the one-loop approximation. The basic procedures can easily be adapted to other theories.

For the purpose of calculating the effective action the observable fields can be treated as the background fields. The unobservable fields are integrated out in the functional path-integral method by use of the steepest-descent approximation. Their effects are felt only through their Green's function, which appeared in the loop integration. The problem of finding the effective-action expansion is reduced to finding solution of the Green's function as a functional of the background fields. The crucial development of this paper is the discovery of the simple formal solution of this Green's function in the momentum space. It has the identical form as the Green's function in the presence of constant background fields except that the background fields are allowed to vary and the argument of the background field is replaced by $x+i\,\partial/\partial p$. The expansion of this formal solution of the Green's function in the power series of $i\,\partial/\partial p$ generates naturally the effective-action expansion.

Now I give the detailed calculation for the self-coupled neutral scalar theory defined by the Lagrangian

$$\mathscr{L}=\tfrac{1}{2}(\partial_\mu\Phi)^2-\tfrac{1}{2}m^2\Phi-V(\Phi)+\text{counterterms}. \quad (1)$$

It is convenient to perform the Wick rotation into the Euclidean space, $X_0\rightarrow -iX_D$ and to adapt the Euclidean metric (1,1,1,1). Throughout this paper, with the exception of the final form of the effective action, the calculation is carried out exclusively in the Euclidean space. The subscripts $(i,j,k, \ldots = 1, \ldots, D$ are used for the Euclidean subscripts. The counterterms are understood to be present in the appropriate places.

The semiclassical approximation is obtained by applying the steepest-descent method to the functional integral

$$Z = N \int D[\Phi] e^{-I[\Phi]/\hbar},$$

with the normalization constant N and the action $I[\Phi]$ defined by

$$I[\Phi] = \int d^D x \{ \tfrac{1}{2}(\partial\Phi)^2 + \tfrac{1}{2} m^2 \Phi^2 + V(\Phi) \}.$$

This action can be expanded around the classical path $\phi(x)$ at which the action is stationary,

$$\Phi(x) = \phi_c(x) + \omega(x),$$

where $\phi_c(x)$ satisfies the equation

$$(-\partial^2 + m^2)\phi_c(x) + V'(\phi_c) = 0.$$

$I[\Phi]$ becomes

$$I[\Phi] = I[\phi_c] + \int d^D x \{ \tfrac{1}{2}\omega[-\partial^2 + m^2 V''(\phi_c)]\omega + \tfrac{1}{6} V'''(\phi_c)\omega^3 + \tfrac{1}{24} V''''(\phi_c)\omega^4 + \ldots \},$$

and the functional integral changes into $\int D[\omega] \ldots$. The effective action $I_{\text{eff}}[\Phi]$ is defined by

$$Z = N \int D[\omega] \exp(-I[\phi_c + \omega]/\hbar) = N' \exp[-\omega(\phi_c)] = N' \exp(-I_{\text{eff}}[\phi]/\hbar)$$

where ϕ is defined by

$$\phi(x) = \frac{\int D[\Phi]\phi(x) e^{-I[\Phi]/\hbar}}{\int D[\Phi] e^{-I[\Phi]/\hbar}} = \phi_c + \frac{\int D[\omega]\omega \exp(-I[\phi_c + \omega]/\hbar)}{\int D[\omega] \exp(-I[\phi_c + \omega]/\hbar)}.$$

This relation can be inverted to give $\phi_c = \phi_c(\phi)$. The resulting effective action is

$$I_{\text{eff}}[\phi] = I[\phi] + \tfrac{1}{2}\hbar \operatorname{Tr} \ln G^{-1} + \tfrac{1}{8}\hbar^2 \int d^D x\, V''''(\phi(x))[G(x,x)]^2$$
$$- \frac{\hbar^2}{12} \int d^D x\, d^D y\, V'''(\phi(x))[G(x,y)]^3 V'''(\phi(y)) + O(\hbar^3), \qquad (2)$$

where the Euclidean Green's function $G(x,y)$ is defined by

$$\{-\partial^2 + U(\phi(x))\} G(x,y) = \delta^D(x-y) \qquad (3)$$

and

$$U(\phi(x)) = m^2 + V''(\Phi(x)). \qquad (4)$$

It is now clear that the semiclassical expansion in the order of h is equivalent to the perturbation expansion in the order of the number of the closed loop. The calculation of the effective action is reduced to solving the Green's function $G(x,y)$ as a function of the background field $U(\phi(x))$ in Eq. (3).

Equation (3) can be solved symbolically,

$$G(x,y) = \int d^D p (2\pi)^{-D} e^{ip\cdot y}[-\partial^2 + U(x)]^{-1} e^{-ip\cdot x},$$

where $U(x) = U(\phi(x))$. Since $[-i\partial_l, x_k] = -i\delta_{lk} \neq 0$, it is not possible to replace $-\partial^2$ by p^2 (unless the field is constant). However, this procedure can be fully justified if x is also replaced by $i\partial/\partial p$ simultaneously while operating on $e^{-ip\cdot x}$,

$$G(x,y) = \int d^D p (2\pi)^{-D} e^{ip\cdot y}[p^2 + U(i\partial/\partial p)]^{-1} e^{-ip\cdot x}.$$

A more useful form can be obtained by use of the property of finite translation from $e^{-ip\cdot x}$,

$$G(x,y) = \int d^D p (2\pi)^{-D} e^{-ip\cdot(x-y)}[p^2 + U(x + i\partial/\partial p)]^{-1}. \qquad (5)$$

Equation (5) is the formal solution of Eq. (3). It is quite apparent that the arguments leading to the solution Eq. (5) are completely independent of the particular form of the Green's function, Eq. (3). Therefore with proper care of the inverse operation the solution Eq. (5) can be generalized to incorporate the spin and internal symmetries by allowing G and U to be matrices in the corresponding spaces,

$$G(x,y) = \int d^D p (2\pi)^{-D} e^{-ip\cdot(x-y)}[G^{-1}(p; U(x + i\partial/\partial p))]^{-1}. \qquad (6)$$

Substituting Eq. (5) into Eq. (2), I obtain the effective Lagrangian through the identification

$I_{\rm eff}[\phi] = -\int d^Dx\,\mathscr{L}_{\rm eff},$

$$\mathscr{L}_{\rm eff}(\phi) = \mathscr{L}(\phi) - \frac{\hbar}{2}\int \delta U(x)\, G(x,x) - \frac{\hbar^2}{2} V''''(\phi(x))[G(x,x)]^2$$
$$+\frac{\hbar^2}{12} V'''(\phi(x))\int \frac{d^Dp}{(2\pi)^D}\frac{d^Dq}{(2\pi)^D}\left[p^2 + U\left(x + i\frac{\partial}{\partial p}\right)\right]^{-1}\left[q^2 + U\left(x + i\frac{\partial}{\partial q}\right)\right]^{-1}$$
$$\times V'''\left(\phi\left(x - i\frac{\partial}{\partial(p+q)}\right)\right)\left[(p+q)^2 + U\left(x - i\frac{\partial}{\partial(p+q)}\right)\right]^{-1} + O(\hbar^3). \quad (7)$$

Manipulations similar to those used for obtaining the formal solution Eq. (5) have been applied for deriving the last term of Eq. (7). I have also used the identity $G(x,x) = \delta\,\mathrm{Tr}\ln G^{-1}/\delta U(x)$.

The effective Lagrangian and therefore the effective-action expansion can be obtained directly by expanding the formal expression of Eq. (7) in power series of the momentum derivative operators such as $i\,\partial/\partial p$. Thus

$$G(x,x) = \int \frac{d^Dp}{(2\pi)^D}\left[p^2 + U\left(+ i\frac{\partial}{\partial p}\right)\right]^{-1}$$
$$= \int \frac{d^Dp}{(2\pi)^D}\frac{1}{p^2+U(x)}\sum_{m=0}^{\infty}\left(-\sum_{n=1}^{\infty}\frac{1}{n!}[\partial_{i_1}\cdots\partial_{i_n}U(x)]\frac{i\partial}{\partial p_{i_1}}\cdots i\frac{\partial}{\partial p_{i_n}}\frac{1}{p^2+U(x)}\right)^m. \quad (8)$$

It should be emphasized here that in this expansion $\partial/\partial p_i$ differentiates the entire expression to its right while the spatial derivative ∂_i only differentiates the $U(x)$ within the same square bracket.

The tensor structure of the momentum integrations can be treated easily by use of the standard symmetry argument. The remaining momentum integrations have the general form of

$$\int \frac{d^Dp}{(2\pi)^d}\frac{p^{2s}}{[p^2+U(x)]^n} = U(x)^{s-n+D/2}\frac{\Gamma(D/2+s)\Gamma(n-s-D/2)}{(4\pi)^{D/2}\Gamma(D/2)\Gamma(n)}.$$

It is more useful to reorder the series in Eq. (8) according to the number of derivatives occurring in each term. After the Wick rotation back to the Minkowski space up to the terms containing four derivatives, $G(x,x)$ becomes

$$G(x,x) = -(4\pi)^{-D/2}\{\Gamma(1-D/2)U^{-1+D/2} - \tfrac{1}{6}\Gamma(3-D/2)U^{-3+D/2}\partial^2 U + \tfrac{1}{12}\Gamma(4-D/2)U^{-4+D/2}(\partial U)^2$$
$$-\tfrac{1}{576}\Gamma(7-D/2)U^{-7+D/2}(\partial U)^4 - \tfrac{1}{120}\Gamma(4-D/2)U^{-4+D/2}\partial^4 U$$
$$+\tfrac{1}{24}\Gamma(6-D/2)U^{-6+D/2}[\tfrac{1}{5}\partial_\alpha U\,\partial_\beta U\,\partial^\alpha\partial^\beta U + \tfrac{1}{6}(\partial^2 U)(\partial U)^2]$$
$$-\tfrac{1}{12}\Gamma(5-D/2)U^{-5+D/2}[\tfrac{1}{12}(\partial^2 U)^2 + \tfrac{1}{5}\partial_\mu U\,\partial^\mu\partial^2 U + \tfrac{1}{15}(\partial_\mu\partial_\beta U)^2]\}. \quad (9)$$

The one-loop corrected effective Lagrangian can now be obtained by substitution of the result of Eq. (9) into Eq. (7):

$$\mathscr{L}_{\rm eff}(\phi) = \mathscr{L}(\phi) + \tfrac{1}{2}\hbar(4\pi)^{-D/2}\{-\Gamma(-D/2)U^{D/2} + \tfrac{1}{12}\Gamma(3-D/2)U^{-3+D/2}(\partial U)^2$$
$$-\tfrac{1}{24}[\tfrac{1}{12}\Gamma(6-D/2)U^{-6+D/2}(\partial U)^4 + \tfrac{1}{5}\Gamma(4-D/2)U^{-4+D/2}(\partial_\alpha\partial_\beta U)^2$$
$$+\tfrac{1}{3}\Gamma(5-D/2)U^{-5+D/2}\partial^\alpha\partial^\beta U\,\partial_\alpha U\,\partial_\beta U]\}, \quad (10)$$

where $U = m^2 + V''(\phi(x))$. Various forms of the four-derivative terms are not independent since they are related through integration by parts. In arriving at Eq. (10) I have chosen to use a canonical form for the $2n$-derivative terms so that for those derivatives operating on the same $U(x)$, (1) their number cannot exceed n, and (2) they cannot contract with each other. If the third and the fourth derivatives on U, such as appeared in Eq. (9), are discarded initially, it would not be possible to arrive at the correct expansion of Eq. (10) even though they do not appear explicitly in Eq. (10).

The divergent parts of Eq. (10) are hidden in the Γ function by the dimension regularization. In a renormalizable theory, $V(\phi)$ should be chosen appropriately for the D dimensions such that the infinite parts can be completely canceled by the counterterms. For the $D=4$ dimension, $V(\phi) = (\lambda/4!)\phi^4$. The finite part $[\hbar/4(4\pi)^2]U^2\ln(U/m^2)$ is the well-known quantum contribution to the effective potential.

$$Z(\phi(x)) = 1 + \frac{\hbar}{12(4\pi)^{D/2}}\Gamma\left(3-\frac{D}{2}\right)U^{D/2-3}(U')^2$$

is in agreement with the result of Iliopoulos, Itzykson, and Martin.

I have presented a systematic method for the effective-action expansion and carried out the explicit calculation for the self-coupled neutral scalar theory. The effective-action expansion of the SU(N) $\otimes$ SU(N) σ model for one quark loop will be presented in another paper in which I propose that such an effective Lagrangian with a single parameter f_π can be used as a realistic model to understand the low-energy dynamics of mesons and baryons.[8] My formulation of the effective-action expansion is a natural extension of the standard functional method approach to the effective potential. The method relies only on the familiar momentum space and Feynman propagator. As long as the preception persists that simplification of fundamental interactions always occurs at shorter distance, this tool will find numerous applications in understanding the low-energy phenomenology.

After this work was completed, I learned that Aitchison and Fraser[10] have also computed the effective expansion for the ϕ^4 theory. Our methods of calculation are completely different.

This work was supported in part by the U. S. Department of Energy.

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