Nonperturbative Effects in Quantum Field Theory on Curved Spaces

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The quantum continuum action for gauge theories in two dimensions is constructed in the axial gauge. For non-Abelian groups the action differs from the usual expectations by nonperturbative, noncoercive corrections, whose role is to ensure gauge invariance for the quantum field theory outside of perturbation theory. Consequently, ordinary perturbation theory and semiclassical approximations become inapplicable. Similar conclusions are expected to apply in more dimensions.

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Consider a classical field theory whose action possesses some symmetry. Suppose we wish to quantize the theory using the path-integral approach in such a way as to preserve as much of the symmetry as possible. (One does not know *a priori* whether spontaneous symmetry breaking will occur or not.) It is generally believed that if the regularization scheme does not manifestly break the symmetry, one must use in the functional integral an action having only symmetric terms with (in general) cutoff dependent coefficients. Except for a few cases for which rigorous proofs exist, this belief is inspired by perturbation theory, where such a procedure produces renormalized Green's functions obeying the correct Ward identities. On the other hand, a counterexample to this rule has been known for many years[1]; for a nonrelativistic particle free to move on the surface of a sphere S^2 the correct path integral for computing the propagator is

$$\langle \Omega_f, T | \Omega_0, 0\rangle = \int D\Omega(t) \exp\{-(1/\hbar)\int_0^T dt[\tfrac{1}{2}\dot{\theta}^2 + \tfrac{1}{2}\sin^2\theta\dot{\vartheta}^2 - \tfrac{1}{8}\hbar^2(1+1/\sin^2\theta)]\}. \tag{1}$$

The quantum action differs from the classical one by a nonspherically symmetric potential term. This term arises from a careful definition of the functional integral and forms the basis of the Itô calculus.[2] It is typical of functional integration on curved spaces.[3]

In a recent paper,[4] Richard and I pointed out that, surprisingly, the presence of the nonsymmetric Itô term in Eq. (1) is intimately related to the insistence that the quantum theory does have spherical symmetry. Indeed, suppose that the infinitesimal kernel were

$$\langle \Omega_{i+1}, \epsilon | \Omega_i, 0\rangle = \frac{1}{2\pi\epsilon}\exp\left\{-\frac{1}{\hbar}\left[\frac{(\theta_{i+1}-\theta_i)^2}{2\epsilon} + \frac{\sin\theta_i \sin\theta_{i+1}(\vartheta_i - \vartheta_{i+1})^2}{2\epsilon}\right]\right\} \tag{2}$$

for $\epsilon \to 0$. By spherical symmetry, the integral

$$I \equiv \int d\Omega_i \langle \Omega_{i+1}, \epsilon | \Omega_i, 0\rangle \tag{3}$$

must be independent of Ω_{i+1} at least up to order ϵ^1. A simple computation[4] yields

$$I = \exp\left[\frac{\hbar\epsilon}{8}\left(1+\frac{1}{\sin^2\theta}\right)\right] + O(\epsilon^2), \tag{4}$$

and proves that the kernel in Eq. (1) is indeed spherically symmetric.

In our paper[4] we also argued that similar nonsymmetric terms should arise in dimensions higher than one in both nonlinear σ models and gauge theories. In the present Letter I would like to illustrate this effect by considering gauge theories in two dimensions. The classical action [for SU(2)] is

$$S_{\rm cl} = -\int d^2x\, \tfrac{1}{4} F^i_{\mu\nu} F^i_{\mu\nu}, \quad F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu - g\epsilon_{ij\beta} A^j_\mu A^\beta_\nu. \tag{5}$$

As a quantum theory, in perturbation theory there are no divergences, hence no nontrivial renormalization of g is to be performed and the action in Eq. (5) is normally supposed to be the correct quantum action. I would like to demonstrate that this is not the case for SU(2) (which is a curved manifold), although it is true for U(1) (which is a flat manifold).

For a gauge invariant definition of the functional integral I shall take Wilson's action[5]:

$$S_{\rm qm} = -(1/2a^2g^2)\sum_{\mathbf{n},\mu\nu} \operatorname{tr} U_\mu(\mathbf{n}) U_\nu(\mathbf{n}+\boldsymbol{\mu}) U^\dagger_\mu(\mathbf{n}+\boldsymbol{\mu}+\boldsymbol{\nu}) U^\dagger_\nu(\mathbf{n}+\boldsymbol{\nu}) + \text{H.c.}, \quad U_\mu(n) = \exp[i\tfrac{1}{2}\boldsymbol{\tau}\cdot\mathbf{B}_\mu(n)]. \tag{6}$$

By use of $\mathbf{B}_\mu(n) = ag\mathbf{A}_\mu(n)$, one easily verifies[6] that for $ag \to 0$

$$S_{\rm qm} = -\tfrac{1}{4}\int d^2x\, F^i_{\mu\nu}F^i_{\mu\nu} + O(a^2g^2). \tag{7}$$

The basis of the analysis to follow, indeed of the Itô calculus, is to show that some of the higher-order terms in Eq. (7) survive in the path integral as $ag \to 0$ and to give a prescription for taking them into account.[7] For this purpose it is convenient to use the axial gauge $\mathbf{A}_\tau = 0$. Equation (6) becomes

$$\begin{aligned} S_{\rm qm} &= -(1/a^2g^2)\sum_{\mathbf{n}} \operatorname{tr} U_x(\mathbf{n})\,U_x^\dagger(\mathbf{n}+\hat\tau) + \text{H.c.} = \sum_x [-(1/a^2g^2)\sum_\tau \operatorname{tr} U_x(\mathbf{n})\,U_x^\dagger(\mathbf{n}+\hat\tau) + \text{H.c.}] \\ &= \sum_x [-(4/a^2g^2)\sum_\tau \mathbf{s}(\mathbf{n})\cdot\mathbf{s}(\mathbf{n}+\hat\tau)], \end{aligned} \tag{8}$$

$$\mathbf{s} \equiv (\cos\tfrac{1}{2}B, \hat{\mathbf{B}}\sin\tfrac{1}{2}B), \quad B \equiv (\mathbf{B}^2)^{1/2}, \quad \hat{\mathbf{B}} \equiv \mathbf{B}/B, \quad 0 \le B \le 2\pi. \tag{9}$$

Equation (8) describes a system of identical uncoupled one-dimensional chains of S^3 spins at temperature $a^2g^2/4$. Computing the expectation value of a Wilson loop in the original gauge theory amounts to computing the spin-spin correlation for the one-dimensional spin model. This is an exactly solvable problem, the answer being

$$\langle \mathbf{s}(0)\cdot\mathbf{s}(m)\rangle = \exp(-\tfrac{3}{8}a^2g^2m). \tag{10}$$

In the coordinates used [Eq. (9)] the partition function we are studying is

$$Z_1 \equiv \left[\prod_i \left(\frac{2}{\pi a^2g^2}\right)^{3/2} \int_{0\le B_i\le 2\pi} \frac{\sin\frac{1}{2}B_i}{2B_i^2}\,d^2B_i\right] \times \exp\{-(4/a^2g^2)[\sum_i \cos\tfrac{1}{2}B_i\cos\tfrac{1}{2}B_{i+1} + \hat{\mathbf{B}}_i\cdot\hat{\mathbf{B}}_{i+1}\sin\tfrac{1}{2}B_i\sin\tfrac{1}{2}B_{i+1}]\}, \tag{11}$$

while the spin-spin correlation function is

$$\langle \mathbf{s}(0)\cdot\mathbf{s}(m)\rangle = \langle \cos\tfrac{1}{2}B(0)\cos\tfrac{1}{2}B(m) + \hat{\mathbf{B}}(0)\cdot\hat{\mathbf{B}}(m)\sin\tfrac{1}{2}B(0)\sin\tfrac{1}{2}B(m)\rangle. \tag{12}$$

In the continuum limit $ag \to 0$, one may be tempted to assume that only quadratic terms in $\mathbf{B}_i - \mathbf{B}_{i+1}$ need to be retained in the exponential in Eq. (11). That is

$$\begin{aligned} \lim_{ag\to 0} Z_1 &\equiv \lim_{ag\to 0}\left[\prod_i\left(\frac{2}{\pi a^2g^2}\right)^{3/2}\int_{0\le B_i\le 2\pi}\frac{\sin\frac{1}{2}B_i}{2B_i^2}d^2B_i\right]\exp\Bigg\{-\frac{4}{a^2g^2}\sum_i\Bigg[\frac{\sin\frac{1}{2}B_i\sin\frac{1}{2}B_{i+1}}{2B_iB_{i+1}}(\mathbf{B}_i-\mathbf{B}_{i+1})^2 \\ &\qquad + \tfrac{1}{2}\hat{\mathbf{B}}_i\cdot(\mathbf{B}_i-\mathbf{B}_{i+1})\hat{\mathbf{B}}_{i+1}\cdot(\mathbf{B}_i-\mathbf{B}_{i+1})\left(\frac{1}{4}-\frac{\sin\frac{1}{2}B_i\sin\frac{1}{2}B_{i+1}}{B_iB_{i+1}}\right)\Bigg]\Bigg\} \\ &= [\prod_i (1/2\pi)^{3/2}\int d^2A_i]\exp[-\sum_i \tfrac{1}{2}(\mathbf{A}_i-\mathbf{A}_{i+1})^2]. \end{aligned} \tag{13}$$

If this expectation were correct, using the Gaussian measure in Eq. (13) to evaluate the spin-spin correlation in Eq. (12) should yield the answer in Eq. (10) for $ag \to 0$. A simple computation shows that in perturbation theory this is the case for U(1), but not for SU(2) starting at $O(a^4g^4)$. An additional test was done numerically, using the Monte Carlo technique.[8] The results are shown in Fig. 1 for $a^2g^2 = \frac{4}{15}$ (expected correlation length 10) on a 52-site lattice (five correlation lengths), with the external spins frozen at $\mathbf{A} = 0$. They show a dramatic discrepancy between the exact result Eq. (10) and its approximation based on the naive quantum continuum action Eq. (13). For comparison, Fig. 1 contains also the results of the computations done for U(1), ($a^2g^2 = \frac{4}{5}$, expected correlation length 10). The exact answer and its Gaussian approximation agree very well.

As expected, retaining only terms of order $(\mathbf{B}_i - \mathbf{B}_{i+1})^2/a^2g^2$ is too naive, because the metric is nonflat. The correct computation could be done following McLaughlin and Schulman.[9] Given the metric for functional integration, they established a procedure for generating the Itô terms. Alternatively, for the case of S^3, one can repeat the arguments and manipulations leading to Eqs. (3) and (4) to obtain the correct (quantum) continuum action to be used in the functional integral. Straightforward, yet lengthy calculations lead to the following possible expres-

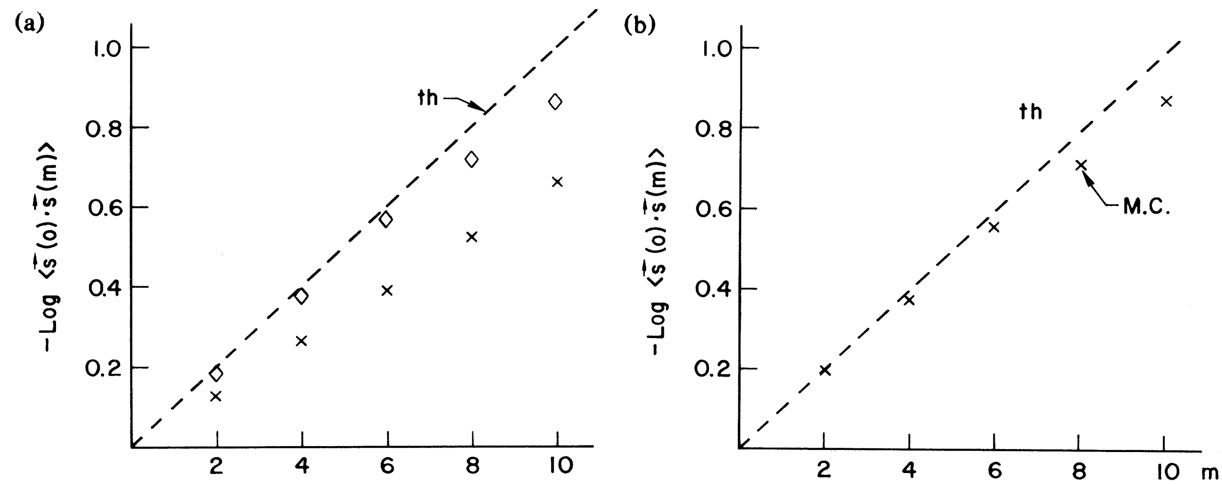

FIG. 1. (a) The spin-spin correlation function for SU(2), and (b) U(1). For SU(2) the Monte Carlo data are shown both for the naive action (crosses) and the Itô action (lozenges).

sions:

$$Z_1=\lim_{ag\to 0}\left[\prod_i\left(\frac{2}{\pi a^2g^2}\right)^{3/2}\int_{0\le B_i\le 2\pi}\frac{\sin^2(\frac{1}{2}B_i)}{2B_i^2}d^2B_i\right]\exp\left\{-\frac{4}{a^2g^2}\sum_i\left[\frac{\sin(\frac{1}{2}B_i)\sin(\frac{1}{2}B_{i+1})}{2B_iB_{i+1}}(\mathbf{B}_i-\mathbf{B}_{i+1})^2\right.\right.$$
$$\left.\left.+\tfrac{1}{2}\hat{\mathbf{B}}_i\cdot(\mathbf{B}_i-\mathbf{B}_{i+1})\hat{\mathbf{B}}_{i+1}\cdot(\mathbf{B}_i-\mathbf{B}_{i+1})\left(\frac{1}{4}-\frac{\sin(\frac{1}{2}B_i)\sin(\frac{1}{2}B_{i+1})}{B_iB_{i+1}}\right)\tfrac{1}{16}a^4g^4V_{B,\mathrm{It\hat{o}}}(B_i)\right]\right\},\qquad(14)$$

$$V_{B,\mathrm{It\hat{o}}}(B)=-\frac{5}{32\sin^2\frac{1}{2}B}-\frac{\cos\frac{1}{2}B}{B\sin\frac{1}{2}B}+\frac{9}{8B^2}+\frac{3B\cos\frac{1}{2}B}{16\sin^3\frac{1}{2}B}-\frac{B\cos\frac{1}{2}B}{16\sin\frac{1}{2}B}-\frac{7}{32},$$

or in spherical coordinates

$$Z_1=\lim_{ag\to 0}\left(\prod_i(2/\pi a^2g^2)^{3/2}\int_0^{\pi}\sin^2\alpha_i d\alpha_i\int_0^{\pi}\sin\beta_i d\beta_i\int_0^{2\pi}d\psi_i\right)$$
$$\times\exp\{-(4/a^2g^2)\sum_i[\tfrac{1}{2}(\alpha_i-\alpha_{i+1})^2+\sin\alpha_i\sin\alpha_{i+1}\tfrac{1}{2}(\beta_i-\beta_{i+1})^2$$
$$+\sin\alpha_i\sin\alpha_{i+1}\sin\beta_i\sin\beta_{i+1}\tfrac{1}{2}(\Psi_i-\Psi_{i+1})^2+\tfrac{1}{16}a^4g^4V_{S,\mathrm{It\hat{o}}}(\alpha_i,\beta_i)]\},\qquad(15)$$

$$V_{S,\mathrm{It\hat{o}}}(\alpha,\beta)=-\frac{1}{8\sin^2\alpha}\left(1+\frac{1}{\sin^2\beta}\right)-\frac{1}{2}.$$

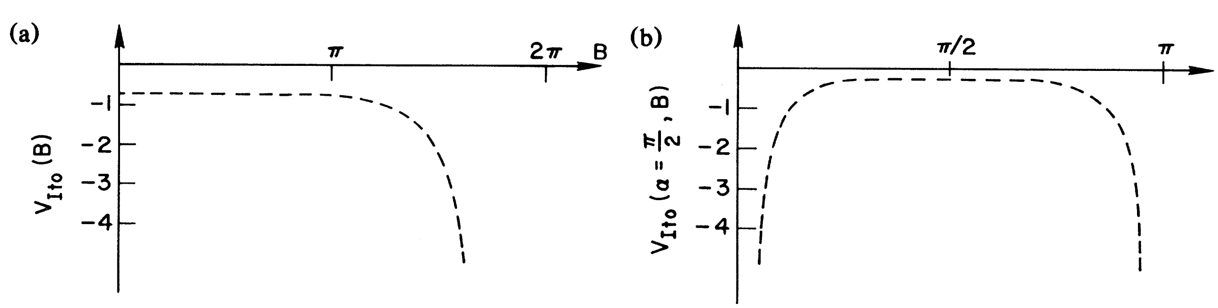

FIG. 2. (a) the Itô potentials $V_{B,\mathrm{It\hat{o}}}(B)$, and (b) $V_{S,\mathrm{It\hat{o}}}(\alpha=\frac{1}{2}\pi,\beta)$.

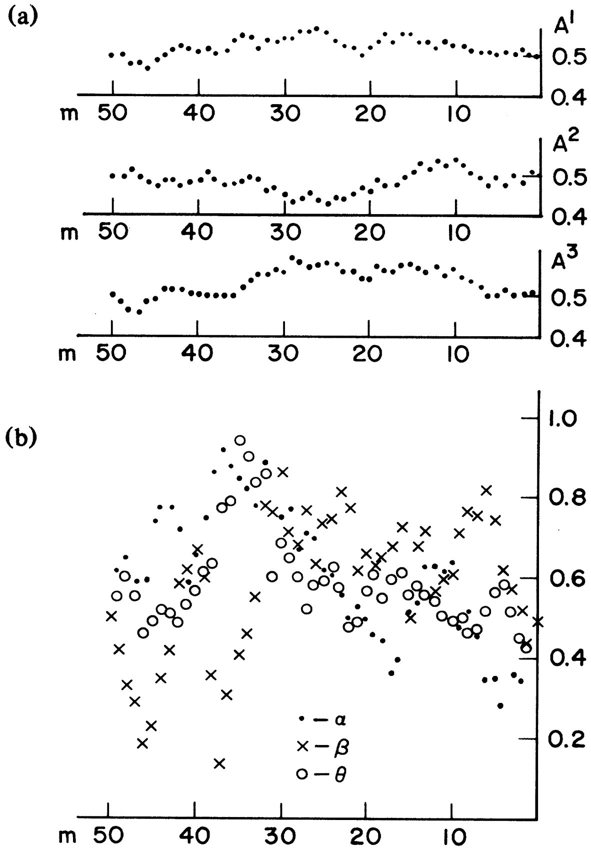

FIG. 3. Some typical spin configurations: (a) with the naive action, and (b) with the Itô action.

Both potentials $V_{B,\mathrm{Itô}}$ and $V_{S,\mathrm{Itô}}$ are noncoercive (see Fig. 2). Their effect is to force the spins to take all values over S^3 with equal probability and restore the spherical symmetry in the functional integral. This fact is illustrated in Fig. 3, where I show some typical (warm) Monte Carlo spin configurations with the naive action of Eq. (13) and the Itô action of Eq. (15). Finally, in Fig. 1 one sees that this latter action produces a spin-spin correlation function in reasonable agreement with the exact dependence.

Conclusions.—1. The expectation that S_{qm} differs from S_{cl} only by the cut-off dependence of certain coefficients is false when the metric of functional integration is nonflat [such as in $O(N)$ $N \geqslant 3$ nonlinear σ models and non-Abelian gauge theories]. Moreover, S_{qm} contains, in general, nonperturbative, noninvariant terms even with an invariant regularization scheme.

2. The presence of the noncoercive Itô terms in S_{qm} makes perturbation theory or semiclassical approximations impossible, since no well-defined regions in field space dominate the functional integral, the field being driven by the Itô potential all over the manifold.[10]

3. Although at the present time the Itô terms can be worked out only in two dimensions, from the analysis in Ref. 4 it seems pretty certain that similar, albeit more drastic, changes of the action occur in more dimensions. These modifications should be of order g^4 and higher.

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