Scaling Behavior of Windows in Dissipative Dynamical Systems

James A. Yorke,[a] Celso Grebogi,[b] Edward Ott,[b],[c] and Laura Tedeschini-Lalli[a]
University of Maryland, College Park, Maryland 20742
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Global scaling behavior for period-n windows of chaotic dynamical systems is demonstrated. This behavior should be discernible in experiments.

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The dependence of the trajectories of nonlinear dynamical systems on a system parameter has recently been the subject of intense interest.[1-3] It is a commonly observed feature of chaotic dynamical systems that, as a system parameter is varied, a stable period-n orbit appears (by a saddle-node bifurcation) which then undergoes a period-doubling cascade to chaos and finally terminates via a crisis[3] (in which the unstable period-n saddle created at the original saddle-node bifurcation collides with the n-piece chaotic attractor; e.g., see Fig. 1 and caption). We call the parameter range between the saddle-node bifurcation and the final crisis a *period-n window.* Perhaps the most widely known example of a window is the period-3 window in the chaotic parameter range of the quadratic map, $x_{n+1}=\mu-x_n^2$ (cf. Fig. 1). In fact, the quadratic map possesses an infinite number of such windows, and these are generally believed to be dense in μ. Furthermore, this situation seems to apply for a wide class of chaotic dynamical systems[4] (including multidimensional systems).

It is the purpose of this Letter to point out and discuss a type of quantitative universal behavior in the global structure of windows. This behavior should be discernible in experiments.

Consider a window and a system parameter μ. Let μ_0, μ_d, and μ_c denote the values of μ at the initial saddle-node bifurcation, the first period-doubling bifurcation, and the final crisis. Then form the normalized crisis value for that window,

$$m_c=(\mu_c-\mu_0)/(\mu_d-\mu_0). \tag{1}$$

Table I gives results for m_c for all primary[5] windows of the map, $x_{n+1}=\mu-x_n^2$, in the range $1.54<\mu<2.00$ that have window width $\mu_c-\mu_0>2.3\times10^{-4}$. Tables II and III give similar results for the Hénon map, $x_{n+1}=1-\mu x_n^2+y_n$, $y_{n+1}=0.3x_n$. Table II applies to all primary windows in the range $1.12<\mu<1.43$ with window widths $\mu_c-\mu_0>2\times10^{-4}$ while Table III is for $1.15<\mu<1.17$ and $\mu_c-\mu_0>5\times10^{-16}$. It is evident from these tables that, for many of the windows, m_c is close to $\frac{9}{4}$. Furthermore, this tendancy increases as the window size becomes smaller. Also, using experimental results supplied to us by P. Linsay, we determine that, for the largest window in Fig. 1 of Brorson, Dewey, and Linsay,[6] $m_c=\frac{9}{4}$ to within the accuracy of the available data ($\pm5\%$). We claim that the following statement applies to any chaotic dynamical process with windows.

1. *Pick any small $\epsilon>0$ and some interval of the system parameter which includes attracting chaotic orbits. Determine the fraction of windows of period $\leqslant n$ in the chosen parameter interval for which m_c deviates from $\frac{9}{4}$ by an amount less than ϵ, $|m_c-\frac{9}{4}|<\epsilon$. Then this fraction will approach 1 as $n\to\infty$, no matter how small ϵ is.*

Several comments are in order concerning Statement 1:

(a) From Tables I and II it appears that the period n does not have to be very large for many of the observed m_c to be close to $\frac{9}{4}$.

(b) The reason for our formulation of Statement 1 in terms of the *fraction* of windows is because, even for arbitrarily high-period windows, there may be some

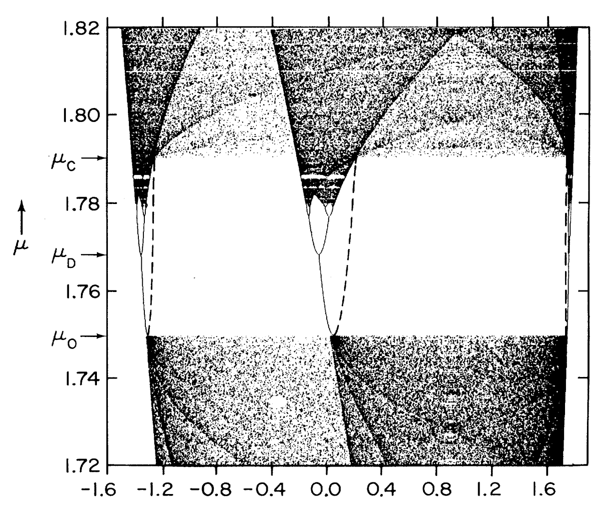

FIG. 1. Computer-generated bifurcation diagram for the map $x_{n+1}=\mu-x_n^2$ in the range $1.72<\mu<1.82$. A period-3 window exists in this range. The window begins with a saddle-node bifurcation (at $\mu=\mu_0\cong1.75$) at which stable and unstable period-3 orbits are born. The unstable orbit is indicated by the dashed line in the figure. The stable period-3 orbit goes through a period-doubling cascade and becomes chaotic as μ is increased. The end of the window occurs at the crisis, $\mu=\mu_c\cong1.79$, when the unstable period-3 created at the original saddle-node bifurcation at $\mu=\mu_0$ (dashed line) collides with the three-piece chaotic attractor.

TABLE I. Results for all primary windows with $\mu_c - \mu_0 > 2.3\times 10^{-4}$ in $1.54 < \mu < 2.00$ for the map $x_{n+1} = \mu - x_n^2$. The last column gives the relative error $|m_c - \frac{9}{4}|/(\frac{9}{4})$.

Window size	Period	μ_0	m_c	Relative deviation of m_c from $\frac{9}{4}$
0.040 327 49	3	1.750 000	2.176	0.033
0.008 961 72	5	1.624 397	2.219	0.014
0.002 211 43	4	1.940 551	2.241	0.0042
0.001 744 14	5	1.860 587	2.241	0.0038
0.001 002 69	7	1.673 954	2.242	0.0036
0.000 389 01	8	1.711 036	2.247	0.0013
0.000 284 60	9	1.595 649	2.248	0.000 88
0.000 264 43	6	1.907 251	2.249	0.000 64
0.000 233 746	9	1.555 257	2.249	0.000 29

few for which m_c deviates substantially from $\frac{9}{4}$ (an indication of this is provided by the period-15 window in Table II).

(c) Statement 1 relates to *global* properties of a window in that m_c is a property of the map for the entire range of μ within the window.

(d) An alternative statement to Statement 1 would be that, within a period-n window, the dynamics generated by the nth iterate of the map, when linearly rescaled, is typically well approximated by the canonical one-dimensional quadratic map,

$$u_{n+1} = u_n + u_n^2 - m, \tag{2}$$

where the form of (2) is such that the original saddle-node bifurcation and the first period doubling occur at $m=0$ and $m=1$, respectively. Thus normalizations, as in (1), are automatic [e.g., for (2) the final crisis is at $m=\frac{9}{4}$].

(e) Remark (d) implies that the choice of the quantity m_c is somewhat arbitrary in that a statement analogous to Statement 1 applies to any other parameter value marking a characteristic event in the window; e.g., replace Eq. (1) by $m_3 = (\mu_3 - \mu_0)/(\mu_d - \mu_0)$ where μ_3 marks the beginning of the period-$3n$ window within the period-n window.

Concerning comment (b), an illustration is instructive. The quadratic map, $x_{n+1} = \mu - x_n^2$, has a period-3 window within its chaotic band (Fig. 1), and the value of m_c for this window is $\frac{9}{4} - 0.074\ldots$. Now consider a period-n window with n large, and assume that (2) provides a good approximation to the dynamics within this window. Clearly, within this window there is a period-$3n$ window with m_c approximately equal to $\frac{9}{4} - 0.074\ldots$. Thus, if one considers the class of windows of period $3n$ which occur as windows within period-n windows, then, for this class, m_c is *not* expected to approach $\frac{9}{4}$ as $n \to \infty$ (rather it is expected to approach $\frac{9}{4} - 0.074\ldots$). However, according to our claim in Statement 1, as n is made large, if one considers *all* orbits of period-$3n$ (not just those which

TABLE II. Hénon-map results for $\mu_c - \mu_0 > 2\times 10^{-4}$ and $1.117 < \mu < 1.427$. The relative error (last column) is $|m_c - \frac{9}{4}|/(\frac{9}{4})$.

Window size	Period	μ_0	m_c	Relative deviation of m_c from $\frac{9}{4}$
0.045 069 88	7	1.226 617	1.635	0.27
0.009 561 20	7	1.299 116	2.032	0.097
0.001 970 90	8	1.121 835	2.229	0.0093
0.000 846 33	9	1.172 384	2.245	0.0022
0.000 492 77	8	1.323 307	2.248	0.000 97
0.000 330 56	10	1.176 765	2.244	0.0027
0.000 322 51	15	1.421 811	1.637	0.27
0.000 301 18	9	1.402 762	2.249	0.000 36
0.000 245 99	13	1.353 915	2.211	0.017
0.000 229 62	10	1.142 882	2.246	0.0018
0.000 229 52	9	1.293 955	2.246	0.0019

TABLE III. Hénon-map results for $\mu_c - \mu_0 > 5\times 10^{-6}$ and $1.15 < \mu < 1.17$.

Window size	Period	μ_0	m_c	Relative deviation of m_c from $\frac{9}{4}$
0.000 080 34	11	1.159 704	2.250 16	0.000 071
0.000 026 07	12	1.150 007	2.249 18	0.000 36
0.000 009 10	16	1.160 453	2.249 69	0.000 14
0.000 008 84	18	1.151 494	2.251 51	0.000 67
0.000 008 71	13	1.155 612	2.250 25	0.000 11
0.000 005 02	16	1.167 380	2.249 83	0.000 074

occur as windows within period-n windows), then the vast majority will have m_c closely approximated by $\frac{9}{4}$ In this respect the type of universal behavior discussed here differs from other previously studied universal behavior.

Because of the situation just described, it appears that it may be difficult to establish Statement 1 in a rigorous way. Thus, at this point, we can only offer arguments that support it but do not prove it. The remainder of this paper is devoted to such an argument for the case of one-dimensional maps a single quadratic maximum. (We emphasize however that, based on Tables II and III, we believe that Statement 1 also applies to multidimensional systems.)

Consider a period-n window with $n >> 1$. At the final crisis for the window there will be n chaotic bands, $S_1, S_2, \ldots, S_n$, each of width $s_1, s_2, \ldots, s_n$, where we have that under the action of the map $S_1 \to S_2 \to \ldots \to S_n \to S_1$, and we choose S_1 to include the critical point, i.e., the maximum of the map function. (There must be one such interval, since otherwise there would be no folding and hence no chaos.) We assume for now that the s_j are small and that for $j \neq 1$ the location of the S_j are sufficiently far from the critical point that the map in the intervals $S_2, S_3, \ldots, S_n$ may be regarded as approximately linear. Let λ_j denote the magnitude of the slope of the map function in the middle of the interval j $(j\neq 1)$. Thus $s_{j+1} \cong \lambda_j s_j$, $n \geq j \geq 2$, and $s_2 \cong K s_1^2$. (For example, $K = \frac{1}{4}$ for $x_{n+1} = \mu - x_n^2$.) Application of these estimates to the entire cycle yields $s_1 = K\lambda^{n-1} s_1^2$, where λ is defined by $\lambda^{n-1} = \lambda_2 \lambda_3 \cdots \lambda_n$, and we call λ the *reduced Lyapunov number*. Thus

$$s_1 \sim \lambda^{-(n-1)}, \tag{3}$$

$$s_j \sim \lambda^{-2(n-1)+(j-2)}, \quad n+1 \geq j \geq 2. \tag{4}$$

Typically we expect λ to be almost constant within the window and larger than 1, reflecting the fact that the orbit, for parameter values outside the window, is chaotic.[7] Within the assumptions above, the n-times iterated map restricted to S_1 can be regarded as the composition of one map, which is quadratic, with $n-1$ approximately linear maps. The result is an approximately quadratic map. The typical closest approach of one of the S_j $(j = 2, \ldots, n)$ to the critical point is $1/n$, which is much greater than s_j since

$$1/n >> \lambda^{-(n-1)}, \tag{5}$$

for large n. As n increases, (5) becomes better and better satisfied, and we expect that the composed map is more and more closely approximated by a quadratic map. Furthermore, the range of μ within the window is small; in fact, as we shall show,

$$\mu_c - \mu_0 \sim \lambda^{-2(n-1)}. \tag{6}$$

Thus the variation of the λ_j with μ can be neglected, and the effect of varying μ is predominantly that of raising or lowering the level of the critical point. Hence the n-times composed map can, under linear rescaling, be put in the form of Eq. (2) and Statement 1 follows. This will be shown in more detail shortly. To see why we must formulate Statement 1 in terms of the fraction of orbits, recall our assumption in the above heuristic argument that the closest approach of S_j $(j = 2, \ldots, n)$ to the origin was $\sim 1/n$. This statement is based on the idea that the orbit for $j = 2, \ldots, n$ is, in some sense, like a chaotic orbit with Lyapunov number λ. According to this point of view, most of the period-n orbits will satisfy our assumption. However, when considering all the $\sim 2^{n-1}/n$ windows of period-n, there is always some "probability" that one of the elements S_j for $n \geq j \geq 2$ will fall too close to the critical point for the linear approximation to hold. As we look at higher n and include more orbits, we should encounter some band orbits of this type.[8]

We now outline more formally how the rescaling yielding Eq. (2) can be obtained. Let $T(x,\mu)$ be a twice differentiable one-dimensional map with a single quadratic maximum (at $x = 0$) and a parameter μ, $x_{n+1} = T(x_n, \mu)$. Assume that T has a period-n window in $\mu_0 < \mu < \mu_c$. At $\mu = \mu_0$ there is a saddle-node bifurcation. Hence $T_x^n(\bar{x}_j, \mu_0) = 1$, where $\bar{x}_j$ are the points of a period-n orbit, $\bar{x}_{j+1} = T(\bar{x}_j, \mu_0)$ with $\bar{x}_{j+n} = \bar{x}_j$, T^n denotes the n-times composed map and $T_x^n = \partial T^n / \partial x$. Let $j = 1$ be chosen so that $\bar{x}_1$ is the closest $\bar{x}_j$ to $x=0$. Define $\tilde{x}(\mu)$ so that $\tilde{x}(\mu_0)=\bar{x}_1$ and $T_x^n(\tilde{x},\mu)=1$. [Note that for $\mu>\mu_0$, $\tilde{x}(\mu)$ is not a member of a period-n orbit.] Let

$$L=T_x(\bar{x}_2,\mu_0)\,T_x(\bar{x}_3,\mu_0)\ldots T_x(\bar{x}_n,\mu_0),$$

where $T_x=\partial T/\partial x$. That is, $L=\lambda^{n-1}$ at $\mu=\mu_0$. Now introduce the rescaling $v=(x-\tilde{x})L$ and $q=(\mu-\mu_0)L^2$. Consider x_{nk} for x_{nk} in the region close to the maximum of T [i.e., $x_{nk}\sim L^{-1}$ (cf. Eq. (3)] and the map $x_{n(k+1)}=T^n(x_{nk},\mu)$. We define a map f, $v_{k+1}=f(v_k,q)$, where v_k corresponds to x_{nk}. Substituting the definitions of v and q into $x_{n(k+1)}=T^n(x_{nk},\mu)$ and expanding for large L we obtain

$$v_{k+1}\cong v_k+\tfrac{1}{2}f_{vv}(0,0)v_k^2+f_q(0,0)q, \qquad (7)$$

where the error in (7) can be shown to be small for large $L=\lambda^{n-1}$ (i.e., large n), provided that points of the x orbit, other than the x_{nk}, do not come too close to $x=0$, as in Eq. (5). Finally, if we set $u=f_{vv}(0,0)v/2$, $m=f_{vv}(0,0)f_q(0,0)q/2$, Eq. (7) becomes Eq. (2). [Note, in addition that the scaling $q=(\mu-\mu_0)L^2$ implies the estimate, Eq. (7).]

In conclusion, we have shown that typical period-n windows exhibit a global scaling structure. Numerically, we find that the scaling is apparent at low n. We have also seen that it is observable in the data of an experiment,[6] and we believe that this behavior should be discernible in experiments generally.

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(a)Department of Mathematics and Institute for Physical Science and Technology.

(b)Laboratory for Plasma and Fusion Energy Studies.

(c)Departments of Physics and of Electrical Engineering.

1J.-P. Eckmann, Rev. Mod. Phys. **53**, 643 (1981); E. Ott, Rev. Mod. Phys. **53**, 655 (1981).

2M. J. Feigenbaum, J. Stat. Phys. **19**, 25 (1978), and *Universality in Chaos,* edited by P. Cvitanovic (Hilger, London, 1984).

3C. Grebogi, E. Ott, and J. A. Yorke, Physica **7D**, 181 (1983), and Phys. Rev. Lett. **48**, 1507 (1982).

4For example, the work of J. A. Yorke and Alligood [Bull. Am. Math. Soc. **9**, 319 (1983), and to be published] implies the creation of an infinite number of windows in the process of forming a horseshoe.

5By a primary window we mean one which is not a window within a window. For example, the period $3\times3=9$ window within the period-3 window (visible, for example, in Fig. 1) is not a primary window. Note that for large n one can show that the fraction of period-n windows which are primary is nearly 1. This follows from the fact that there are of the order of $2^{n-1}/n$ period-n windows. Thus, for example, the fraction of period-35 windows that occur as period-5 windows inside period-7 windows is of the order of $(2^4/5)(2^6/7)/(2^{34}/35)=2^{-24}$.

6S. D. Brorson, D. Dewey, and P. S. Linsay, Phys. Rev. A **28**, 1201 (1983).

7For a narrow window the bands S_j are narrow and their locations vary little over the range of μ in the window. Thus for $j=2,3,\ldots,n$ the λ_j are almost constant, and the *reduced* Lyapunov number is almost constant. In contrast, the usual Lyapunov number which takes into account the whole orbit (*including* its elements in S_1) varies widely over the window.

8In the light of Eq. (6), we offer the following comment concerning Table I. In Table I the windows are arranged in order of decreasing window size (leftmost column), and we see that the relative deviation of m_c from $\frac{9}{4}$ (rightmost column) decreases regularly through two orders of magnitude. Thus the table suggests that smaller window size implies smaller deviation (as do our heuristic arguments). Coupled with (6), the Table I result implies that higher-period windows tend to have smaller deviations.