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Energy-Density Correlation Functions in the Two-Dimensional Ising Model with a Line Defect

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We consider the two-dimensional Ising model on a rectangular lattice with one row of horizontal coupling strengths changed. We shall present our results for correlation functions with two energy-density operators in arbitrary positions, restricting ourselves here to the scaling limit. These exact results show an unexpected complexity.

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The two-dimensional Ising model with a line defect was studied by many authors.[1–4] The model is characterized by the fact that the interaction constants for the nearest-neighbor pairs on this line are changed by an amount ΔJ with respect to their bulk value. Fisher and Ferdinand[1] found that the incremental specific heat due to this line defect diverges linearly as $T\to T_c$, so that the surface exponent $\alpha_s=1$ satisfies the scaling relation[1,5] $\alpha_s=\alpha_b+1$, α_b being the bulk specific-heat exponent.

It was later shown by Bariev[2] and McCoy and Perk[3] that the local magnetization near this line has a continuous exponent $\beta_\sigma=\beta(\Delta J)$, that the local disorder above T_c has exponent[3] $\beta_\mu=\beta(-\Delta J)$, and that the correlation of two spins on this line at T_c decays with a power law with distance having exponent[3] $\eta_\sigma=2\beta(\Delta J)$. Bariev attributed the nonuniversal behavior to the existence of a marginal operator.[2,6,7] In fact, the perturbation of the Hamiltonian due to the defect,

$$\Delta\mathscr{H}=-\Delta J\sum\nolimits_n\sigma_{0,n}\sigma_{0,n+1}=-\Delta J\sum\nolimits_n\epsilon_{0,n}, \qquad (1)$$

has dimension $d_{\rm per}=1$. The scaling dimension $\Delta_{\rm per}$ of the energy-density operator $\epsilon_{0,n}$ is also 1, so that the condition for marginality $\Delta_{\rm per}=d_{\rm per}$ holds.

However, when the line defect is periodically repeated in the lattice—a special case of the "layered Ising models"—the dimension of the perturbation is $d_{\rm per}=2\neq 1=\Delta_{\rm per}$. Then one does not expect nonuniversal behavior, in agreement with all known results[8] on layered Ising models for which the exponents are Ising type, as long as the periods are finite.[9]

Also, for the semi-infinite Ising lattice, even if the spins on the boundary have modified couplings,[10] the surface magnetization has the usual and therefore universal surface exponent[11] $\beta_s=\frac{1}{2}$. This can be understood from the fact[12] that near the boundary the energy-density two-point correlation function decays with distance as R^{-4}. So, the scaling dimension of the perturbation is $\Delta_{\rm per}=2$, which is not equal to the dimension of the perturbation $d_{\rm per}=1$. The different values for the scaling dimension $\Delta_{\rm per}$ of the energy-density operator, bulk versus boundary, may explain the phenomenon that weakening of bonds on the boundary induces a "roughening transition" whereas weakening of bonds in the interior does not.[13] If, however, one adds a nearest-neighbor coupling of strength inversely proportional to the distance from the boundary to the semi-infinite Ising lattice, one introduces a marginal operator ($d_{\rm per}=2$) and continuous exponents have been found in this case.[14]

It has been shown[7] that even though the line defect introduces a marginal operator into the system, the exponent of the energy-density correlation is universal, i.e., it remains at its bulk value.[15] This is in agreement with the result of Fisher and Ferdinand[1] that the incremental specific heat, which is expressible as a sum over energy-density correlations, has a universal exponent $\alpha_s=1$. This does not mean that the energy-density correlations in the line-defect model are without much structure. On the contrary, the scaling-limit results that we shall present have a rich behavior and may be well worthy of further study by other methods.

For instance, recently the conformal algebra approach to critical phenomena has been used with great success,[16] explaining critical exponents for many two-dimensional systems and giving results for scaled critical-point correlation functions. Most of these calculations have been done for bulk properties, with the assumption of translational and rotational invariance from the outset. Cardy,[17] however, has considered a semi-infinite system with a free boundary. He found that the n-point correlation functions in the semi-infinite system are related to $2n$-point correlation functions of the bulk system, involving the n original points and their n "mirror charges." He came to this conclusion after comparing the sets of linear differential equations implied by the conformal invariance and imposing suitable boundary conditions. The line-defect model interpolates between the semi-infinite model and the bulk system. It would be most interesting, though not entirely trivial, to extend the conformal algebra method also to this line-defect model.

We shall now summarize our scaling-limit results for the energy-density correlation function. Details of the derivation and full results for arbitrary temperature shall be presented elsewhere, including implications for one-dimensional models.[18]

We consider the two-dimensional Ising system defined by the reduced Hamiltonian

$$\mathscr{H}/kT = -\sum_{m,n=-\infty}^{\infty} (K_1\sigma_{m,n}\sigma_{m,n+1} + K_2\sigma_{m,n}\sigma_{m+1,n}) - \lambda \sum_{n=-\infty}^{\infty} \sigma_{0,n}\sigma_{0,n+1}, \tag{2}$$

where kTK_1 and kTK_2 are the horizontal and vertical couplings between the nearest-neighbor pairs. In the center row the horizontal couplings are changed by the amount $\Delta J = kT\lambda$. (Without loss of generality, we can assume $K_1, K_2 > 0$.)

The canonical average $\langle \sigma_{m,0}\sigma_{m,1}\sigma_{l,n}\sigma_{l,n+1}\rangle$ has been calculated exactly, by use of the transfer matrix method, in terms of a 4×4 Pfaffian whose elements are single integrals. We have analyzed this result in the scaling limit. We find that the scaling function is not rotationally invariant, in contrast to the one for the pure system ($\lambda = 0$), for which[15]

$$\langle \sigma_{0,0}\sigma_{0,1}\sigma_{m,n}\sigma_{m,n+1}\rangle - \langle \sigma_{0,0}\sigma_{0,1}\rangle^2 \simeq (\pi\xi_h)^{-2}[K_1^2(r) - K_0^2(r)], \tag{3}$$

where $K_0(r)$ and $K_1(r)$ are modified Bessel functions and r is the scaled distance

$$r = [(n/\xi_h)^2 + (m/\xi_v)^2]^{1/2}. \tag{4}$$

Here, the horizontal and vertical correlation lengths are given by

$$\xi_h^{-1} = 2|K_1^* - K_2|, \quad \xi_v^{-1} = 2|K_2^* - K_1|, \tag{5}$$

with K_1^* and K_2^* being the vertical and horizontal (reduced) couplings of the dual lattice,

$$\sinh(2K_\alpha)\sinh(2K_\alpha^*) = 1 \quad (\alpha = 1,2). \tag{6}$$

The critical point $T = T_c$ is given by $K_1^* = K_2$ $(K_2^* = K_1)$.

To shorten the notation, we denote the horizontal and vertical energy-density operators by

$$\delta\epsilon_{m,n}^h = \sigma_{m,n}\sigma_{m,n+1} - \langle \sigma_{m,n}\sigma_{m,n+1}\rangle, \quad \delta\epsilon_{m,n}^v = \sigma_{m,n}\sigma_{m+1,n} - \langle \sigma_{m,n}\sigma_{m+1,n}\rangle. \tag{7}$$

The scaling functions for the horizontal and vertical energy-density two-point functions are the same. They can both be written in terms of the integral

$$F_\tau(x,y) = \int_0^\infty dq\, \exp[-y(1+q^2)^{1/2}]\cos(xq)/(1+q^2)^{1/2}[(1+q^2)^{1/2} - \tau], \tag{8}$$

where

$$\tau \equiv \tanh(2\lambda)\,\mathrm{sgn}(T - T_c). \tag{9}$$

One can easily verify that

$$[\partial/\partial y + \tau]F_\tau(x,y) = -K_0(r), \quad r \equiv (x^2+y^2)^{1/2}, \tag{10}$$

$$[\partial^2/\partial x^2 + \partial^2/\partial y^2 - 1]F_\tau(x,y) = 0. \tag{11}$$

Then, if the two-energy density operators are on opposite sides of the defect line, we find in the scaling limit

$$(\pi\xi_\alpha)^2\langle\delta\epsilon^\alpha_{l,0}\delta\epsilon^\alpha_{m,n}\rangle=(1-\tau^2)\left\{\left[\frac{\partial^2}{\partial x\,\partial y}F_\tau(x,y)\right]^2+\left[\frac{\partial^2}{\partial y^2}F_\tau(x,y)\right]^2-\left[\frac{\partial}{\partial y}F_\tau(x,y)\right]^2\right\} \tag{12}$$

(for $\alpha=h,v$ and $l\leqslant 0\leqslant m$). Here, x and y are the scaled distances between the two spin pairs,

$$x=n/\xi_h,\quad y=(m-l)/\xi_v. \tag{13}$$

If both operators are on the same side of the defect line, we have

$$(\pi\xi_\alpha)^2\langle\delta\epsilon^\alpha_{l,0}\delta\epsilon^\alpha_{m,n}\rangle=\left[\frac{\partial}{\partial x}K_0(r)+\tau\frac{\partial}{\partial x}F_\tau(x,\bar{y})\right]^2+\left[\frac{\partial}{\partial y}K_0(r)\right]^2-\left[\tau\frac{\partial^2}{\partial x\,\partial\bar{y}}F_\tau(x,\bar{y})\right]^2-\left[K_0(r)+\tau\frac{\partial^2}{\partial x^2}F_\tau(x,\bar{y})\right]^2 \tag{14}$$

(for $\alpha=h,v$ and $0\leqslant l\leqslant m$). Here x and y are defined by (13) and

$$\bar{y}=(m+l)/\xi_v,\quad r=(x^2+y^2)^{1/2}. \tag{15}$$

For $\Delta J=\lambda=\tau=0$, Eqs. (12) and (14) reduce to the bulk result (3) by use of (10). Also, if $\bar{y}\gg y$ in (14), meaning that both pairs are on the same side of and far from the defect line, we recover the bulk result (3). This follows from Eq. (8), when we note that $F_\tau(x,\bar{y})\propto e^{-\bar{y}}\to 0$, for $\bar{y}\to\infty$. Finally, if l is set equal to 0, the two expressions (12) and (14) can be shown to be identical by use of Eqs. (10) and (11).

The line-defect model (2) is transformed by the duality transformation [see Eq. (6)] into an Ising model with one row of vertical couplings modified. Therefore, we also have the results for the "ladder model," which contains the semi-infinite Ising lattice with a free boundary as the limit $\lambda\to\infty$. One can easily verify that in this limit Eq. (14) reduces to the result of Bariev.[12]

If the distance between the energy-density operators is much larger than the correlation length, we can obtain the asymptotic behavior by substituting into (12) and (14) the asymptotic expansion

$$F_\tau(x,y)\simeq\pi(1-\tau^2)^{-1/2}\exp[-y\tau-x(1-\tau^2)^{1/2}]\Theta(r\tau-y)+\frac{(\pi r/2)^{1/2}e^{-r}}{(y-r\tau)}\left\{1-\frac{8r^2-3y^2-6yr\tau+r^2\tau^2}{8r(y-r\tau)^2}+O(r^{-2})\right\}, \tag{16}$$

for $r\to\infty$, $\Theta(x)$ being the Heaviside step function. The apparent discontinuity on the lines $y=r\tau$ is the well-known "Stokes phenomenon" for asymptotic expansions. A more careful analysis shows that the two different regimes are smoothly connected through an error function. Since $y\geqslant 0$, the step function can only contribute for $\tau\geqslant 0$.

For $\tau<0$, the energy-density two-point correlation decays as e^{-2r}/r^2, just like the bulk result (3), but now with a coefficient which is angle dependent. If $\tau>0$, this form of decay persists provided $y>r\tau$ in (12), or $\bar{y}>\bar{r}\tau$ in (14), $\bar{r}=(x^2+\bar{y}^2)^{1/2}$. But for $y<r\tau$, or $\bar{y}<\bar{r}\tau$, there is a crossover to the Ornstein-Zernike–type behavior

$$e^{-\kappa r}/r^{1/2},\quad \kappa=1+\tau(y/r)+(1-\tau^2)^{1/2}(x/r). \tag{17}$$

From the definition (9), we see that this anomalous behavior only occurs if either $\Delta J>0$ and $T>T_c$ or $\Delta J<0$ and $T<T_c$.

Finally, it is particularly interesting to consider the limit $r\ll 1$, for which the separation of the energy-density operators is much smaller than the correlation length. From (8) we then have $\partial F_\tau(x,y)/\partial y\simeq -\ln r$, and, also using (10) and (11), we find that the leading contributions to (12) and (14) are given by

$$\langle\delta\epsilon^h_{l,0}\delta\epsilon^h_{m,n}\rangle\simeq\mu^2\langle\delta\epsilon^v_{l,0}\delta\epsilon^v_{m,n}\rangle\simeq\begin{cases}\dfrac{1}{(\pi R\cosh 2\lambda)^2}, & l\leqslant 0\leqslant m,\\[2ex] \dfrac{1}{(\pi R)^2}-\dfrac{(\tanh 2\lambda)^2}{(\pi\bar{R})^2}, & l,m\geqslant 0,\end{cases} \tag{18}$$

where

$$R = [n^2 + \mu^2(m-l)^2[^{1/2},$$
$$\bar{R} = [n^2 + \mu^2(m+l)^2]^{1/2}, \tag{19}$$
$$\mu = \xi_h/\xi_v = \sinh(2K_{1c}).$$

This is also precisely the critical-point result as can be shown by a separate calculation. The result (19) shows a striking similarity with the electrostatics problem of two point charges near a boundary separating two media with different dielectric constants, commonly solved by the method of images.

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