Scaling in Spin-Glasses

Daniel S. Fisher
AT&T Bell Laboratories, Murray Hill, New Jersey 07974

and

H. Sompolinsky
Department of Physics, Bar Ilan University, Ramat Gan, Israel 52100
(Received 29 October 1984)

Many results of the mean-field theory of spin-glasses violate simple scaling laws, including the magnetic field dependence of the transition lines, crossover effects of random anisotropy, and critical behavior in the ordered phase. These violations arise from two dangerously irrelevant variables. Below *eight* dimensions some mean-field exponents change and at $d=6$ scaling is restored. Below $d=6$ all scaling laws should be valid implying that the critical properties of *real* spin-glasses should be simpler than in mean-field theory. Experimental consequences are discussed.

PACS numbers: 75.40.Bw

The nature and even the existence of the spin-glass (SG) transition in three-dimensional systems with short-range interactions are currently debated. Frequently, these questions have been studied, experimentally[1-5] as well as theoretically,[6-9] from a phenomenological scaling approach. This assumes that singularities which appear in static and dynamic quantities as the SG transition is approached both from *above* and from *below* can be described by simple scaling laws as for ordinary critical points.

A great deal of theoretical information about the SG transition has been provided by the mean-field (MF) theory which is expected to be valid in high dimensions.[10,11] However, many results of the MF theory violate the ordinary scaling laws, especially those involving the behavior *below* the transition temperature. One expects generally that hyperscaling will be violated above the upper critical dimension, d_c.[12] However, scaling laws which do not explicitly involve the dimensionality d are usually obeyed in *all* numbers of dimensions. Moreover, mean-field theories are usually consistent with the scaling laws involving d at d_c. However, for spin-glasses, violations of scaling in the MF theory seem to be rather more serious. They involve scaling relations which do not explicitly depend on d. In particular, the usual scaling relation between magnetic field and temperature is not obeyed and there are apparently two order-parameter exponents. Furthermore, the MF results are not consistent with all the expected scaling relations involving d (in particular the Josephson relation) at $d_c=6$ which is the upper critical dimension of the SG transition.

Understanding the origins of the scaling violations in MF theory is important in light of the extensive efforts that have been invested in the comparing of experimental results on three-dimensional systems with MF theory. In particular, it is essential to know whether scaling violations are a general property of spin-glass transitions or anomalies of the MF limit. If the latter is correct, then we must ask what are the appropriate scaling laws for SG transitions in, for example, three-dimensional systems with short-range interactions. From a theoretical point of view, understanding the scaling behavior of the SG critical fixed point in high dimensions in a prerequisite for performing renormalization group calculations of the effects of fluctuations on SG transitions.

In this paper, we show that the violations of scaling in MF theory are the result of the existence of two dangerously irrelevant variables. We argue that the anomalous MF results apply *only* for $d>8$. For $6<d<8$, the exponents are modified but can still be determined exactly. As d approaches 6, the normal scaling laws are restored and for $d<6$ they should hold. We thus expect that the SG transition in three dimensions should obey ordinary scaling laws. At the end of this paper, we summarize the consequences of our results for the scaling behavior of SG transitions in three-dimensional systems with short-range interactions; these should be valid as long as the SG transition temperature is nonzero.

The simplest violation of scaling in MF theory is the behavior of the phase diagram in a weak magnetic field, h. Above the zero-field transition temperature, T_g, the crossover from the zero-field regime to the noncritical (paramagnetic) regime occurs at a reduced temperature $t\equiv T/T_g-1$ which is proportional to $h^{2/\Delta}$ with the exponent Δ of the ordering field h^2 equal to 2. This crossover can be seen, for example, in the nonlinear magnetic susceptibility which is proportional to the spin-glass susceptibility.[13] The value $\Delta=2$ is consistent with the scaling law $\Delta=1-\alpha/2+\gamma/2$, involving the MF exponents (defined above T_g) $\gamma=1$ and $\alpha=-1$. On the other hand, the SG transition in a finite field occurs in an Ising system at $t_c(h)\sim -h^{\theta_I}$ with $\theta_I=2/3$ [the Almeida-Thouless (AT) line].[14] This is in contrast to the behavior which is expected from scaling: $\theta_I=2/\Delta$. This anomaly of the MF theory has been noted previously and has been attributed to special cancellations.[3] We now analyze how this and other anomalous scalings arise from a renormalization group point of view and show how they are modified by fluctuations below $d=8$.

It is convenient to work with a replicated Ginzburg-Landau-Wilson effective Hamiltonian.[15] This Hamiltonian is expressed in terms of the order-parameter matrix $Q^{\alpha\beta}$, the thermal average of which is $\langle S^\alpha S^\beta\rangle$ where S^α and S^β are spins belonging to two different replicas. For an Ising spin-glass (which we discuss here for simplicity) we have (dropping unimportant terms)

$$H=\int d^dx\{\sum[\tfrac{1}{4}t(Q^{\alpha\beta})^2+\tfrac{1}{4}(\nabla Q^{\alpha\beta})^2] - w\sum Q^{\alpha\beta}Q^{\beta\gamma}Q^{\gamma\alpha}+y\sum(Q^{\alpha\beta})^4\} \quad (1)$$

where the sums run over replica indices 1 to n with the restriction that $Q^{\alpha\beta}=0$ for $\alpha=\beta$ and the limit $n\rightarrow 0$ is taken. The bare value of the coefficient y is *negative* and that of w is positive. In more than six dimensions, w and y are irrelevant at the critical fixed point which occurs at $t=0$, and thus the upper critical dimension is six.

Normally, scaling functions can be derived by setting the irrelevant variables equal to 0. However, if a quantity depends in a singular fashion on an irrelevant variable, then this cannot be done and the result is that scaling relations between exponents can be violated. Such irrelevant variables are said to be *dangerously irrelevant*.[16] The best known example is the coefficient, u, of the quartic term in a ϕ^4 theory above four dimensions. The singular part of the free energy of this model below T_c is inversely proportional to u which causes the breakdown of hyperscaling for $d>4$, i.e., $d\nu\neq 2-\alpha$. Scaling laws not involving d are still obeyed, however.

For SG's in $d>6$, w is similarly dangerously irrelevant and causes a breakdown of hyperscaling. However, because of cancellations of the effects of the cubic term in Eq. (1) in the limit $n\rightarrow 0$, the more strongly irrelevant fourth-order term determines many of the properties of the low-temperature phase and the spin-glass transition line as a function of magnetic field so that y is also dangerously irrelevant.

We now consider the effects of critical fluctuations for $d>6$ and show that they modify some of the exponents already for $d<8$. The only possible source of corrections to the exponents for $d>6$ is the renormalization of y by four cubic vertices (a "box" diagram). The differential recursion relation for y is (dropping unimportant terms)

$$dy/dl=(4-d)y-Aw^4, \quad (2)$$

where A is a *positive* coefficient so that y is always negative. By use of $dw/dl=\frac{1}{2}(6-d)w$ and integration of Eq. (2) it can be seen that for $6<d<8$ the second term controls the long-distance behavior of y so that $y(l)\approx -w^4(l)$.

The critical behavior of various quantities can be computed by integration of the recursion relations $dt/dl=2t$, $dw/dl=\frac{1}{2}(6-d)w$, $dh^2/dl=(1+d/2)h^2$, and Eq. (2), until $t(l^*)=te^{2l^*}$ is of order 1 and then using the mean-field results with $w(l^*)$ and $y(l^*)$ which should be valid since the Hamiltonian at scale e^{l^*} is far from critical. For example, for $6<d<8$ the temperature dependence $h_{\rm AT}(t)$ of the Almeida-Thouless line [i.e., the inverse of $t_c(h)$] is given by

$$h_{\rm AT}^2(t,w,y)\sim e^{-(1+d/2)l^*}h_{\rm AT}^2(t(l^*),w(l^*),y(l^*)) \sim |t|^{1/2+d/4}\frac{y(l^*)}{w(l^*)^3}t(l^*)^3\sim |t|^{d/2-1}, \quad (3)$$

yielding $\theta_I=4/(d-2)$. For $d\rightarrow 8$ this approaches the MF result which will hold for all $d>8$. At the upper critical dimension $d_c=6$, we have $\theta_I=1$ (up to possible logarithms) which agrees with the scaling of h with t above T_g as expected: i.e., $\theta_I=2/\Delta$. The modification of θ_I in Eq. (3) was proposed by Green, Moore, and Bray[17] on the basis of a one-loop calculation. Here, we argue that this is exact for $6<d<8$ and part of a more general modification of the MF behavior. All of the quantities which scale anomalously in MF theory are proportional to powers of y and thus their behavior will be modified for $d<8$.

In the manner illustrated above, we derive the following modifications of the MF results for $6<d<8$[18]:

(1) *Scaling with respect to h and t.*—In the Ising case the transition line has the form $|t_c|\sim h^{4/(d-2)}$ (for $h\rightarrow 0$) as shown above. For the Heisenberg case in MF theory, on the other hand, $|t_c|\sim h^{\theta_H}$ with $\theta_H=2$ (the Gabay-Toulouse line[19] which has incorrectly been interpreted as an analytic correction[20]). Again, integrating the recursion relations leads to $\theta_H=4/(10-d)$. This agrees with the MF result for $d=8$, but yields $\theta_H=1$ for $d=6$, which is consistent with scaling since for $d>6$, $\Delta=2$, also for the Heisenberg case.

(2) *Random anisotropy.*—The presence of a small anisotropy, D, which couples randomly all spin components in a Heisenberg spin-glass leads to Ising critical behavior.[21,22] This manifests itself as a crossover of the finite-field transition from Heisenberg to Ising character. One would generally expect from scaling that this crossover occurs at $|t|\sim D^{2\phi_A}$ with $\phi_A=1$ for $d\geqslant d_c$. In contrast, MF theory predicts that Ising behavior occurs for $|t|$ less than a characteristic scale $t_I\sim D^{2/\phi_I}$, with $\phi_I=2$, whereas Heisenberg behavior is seen only for $|t|$ greater than $t_H\sim D^{2/\phi_H}$ with $\phi_H=\frac{5}{2}$.[22] Using $dD^2/dl\sim 2D^2$ which holds for $d>6$,

we find that for $6 < d < 8$ fluctuations modify the crossover exponents to $\phi_H = (3d-14)/4$ and $\phi_I = (d-4)/2$. Both crossover exponents thus reduce to $\phi_A = 1$ for $d = 6$.

(3) *Critical properties below T_g.*—Studies of critical fluctuations around the MF theory at high d,[23,24] as well as to leading order in $6-d$,[15] show that the exponents of the SG transition in zero field *when approached from above* obey the usual scaling laws except for the violation of hyperscaling for $d > 6$. However, the critical properties of the transition when approached from the SG phase are more complex. Expanding around MF theory[21,24] yields two-point correlation functions which decay with two different correlation lengths: One diverges with an exponent $\nu = \frac{1}{2}$ as above T_g and the second is a much longer length with $\tilde{\nu} = 1$. In addition, the MF theory yields[10,11] an infinite number of order parameters $Q^{\alpha\beta}$ for the SG phase, which can be parametrized by an order function $q(x)$ with $0 \leqslant x \leqslant 1$. Since all the order parameters are two-spin operators with the same symmetry, scaling would imply that all combinations of them have the same critical behavior. Instead, one finds that the Edwards-Anderson order parameter $q_{\rm EA} = q(1)$ goes to 0 with an exponent $\beta_q = 1$ which is consistent with the scaling law $\beta_q = (2-\alpha-\gamma)/2$ involving the exponents above T_g. On the other hand, the irreversible response,[11,25] $\Delta(0) = [q(1) - \int_0^1 q(x)dx]/T$, which determines the difference between the ac and the equilibrium magnetic susceptibilities goes to zero with an exponent $\beta_\Delta = 2$. The correlation length ξ which diverges as $|t|^{-1/2}$ below T_g does not depend singularly on y. However, the exponent $\tilde{\nu} = 1$ is associated with a correlation length $\tilde{\xi}$ which diverges (at fixed t) as $y \rightarrow 0$. We find therefore that for $6 < d < 8$, $\tilde{\nu}$ is modified to $\tilde{\nu} = d/4 - 1$. As d approaches 6, $\tilde{\nu} \rightarrow \nu = \frac{1}{2}$. Similarly, replica symmetry breaking disappears if $y \rightarrow 0$, and therefore the exponents of quantities which are associated with it will be modified. Using $dQ^{\alpha\beta}/dl = (d/2-1)Q^{\alpha\beta}$ for all $\alpha\beta$, one obtains, for example, that the irreversible response exponent is $\beta_\Delta = d/2 - 2$ which approaches $\beta_q = 1$ as $d \rightarrow 6$.

(4) *Exchange stiffness constant.*—For the isotropic Heisenberg spin-glass, there is a spin-wave stiffness constant ρ_s below T_s which vanishes at T_g as $\rho_s \sim |t|^\mu$, with $\mu = 3$.[26] On the basis of scaling one expects a Josephson relation $\mu = (d-2)\nu$ which should hold for any $d \leqslant d_c$. But the result $\mu = 3$ is inconsistent with this relation for $d = d_c = 6$. In a SG phase with no replica symmetry breaking (e.g., in the limit of infinite number of spin components), ρ_s given by

$$\rho_s \propto [q^2(1) - \int_0^1 q^2(x)dx] \propto (y/w^4)|t|^3$$

would vanish. Thus, although $\rho_s(l) \sim \rho_s e^{(d-2)l}$, the Josephson relation breaks down *above* $d = 8$ because of the singular dependence on y. However, below $d = 8$, $y(l)/w^4(l)$ tends to a constant and therefore the Josephson relation $\mu = (d-2)\nu$ is restored, yielding $\mu = 2$ instead of 3 in $d = 6$.

We now discuss the expected scaling behavior near SG transitions of three-dimensional systems with short-range interactions, assuming T_g is nonzero. We have shown that the anomalous properties of the MF theory from the point of view of scaling are particular to spin-glasses in many dimensions and disappear as d approaches $d_c = 6$. In *fewer* than six dimensions, w will have a nonzero fixed-point value and hence so will y. Their *deviations* from their fixed-point values will not be dangerously irrelevant. Thus, while the exponents will change from their values at $d = 6$, the scaling relations between them should be preserved and the SG transition should obey the usual scaling laws. In particular, the exponent $2/\theta$ of the finite-field transition lines, if they exist, should be equal below $d = 6$, to the ordering-field exponent $\Delta = 1 - \alpha/2 + \gamma/2 = \nu(d+2-\eta)/2 = \gamma + \beta$ which determines the crossover in the nonlinear magnetic susceptibility above T_g. The equality $\Delta = 2/\theta$ should hold for *both* Ising and Heisenberg spin-glasses as demonstrated by our results for $d \rightarrow 6^+$. This is in contrast to recent claims in the literature.[27] It should be noted, however, that for d less than 6, the exponents for Heisenberg and Ising systems will generally differ. Recent experiments[28] on GdAl yield for both Δ and $2/\theta$ the value $\Delta \approx 3.3$. This value is consistent with the behavior $|t_c| \sim h^{2/3}$ for the AT-like finite-field transition line found in many spin-glasses.[3,25,28,29] It is also similar to recent numerical estimates[8,30] for a three-dimensional Ising SG which yield $1.3 \gtrsim \nu \gtrsim 1.1$ and $\eta \sim 0$, implying $\Delta \approx 3$. Although the bulk uniform anisotropy in GdAl (and other nominally Heisenberg SG's) is small and it might thus be expected to exhibit Heisenberg behavior, the random anisotropy may be sufficiently strong to lead to Ising behavior. At this stage it is unclear whether the apparent agreement between the measured[28] Δ and the Ising simulations[30] is due to the effects of the random anisotropy or to a presumably fortuitous similarity between the Heisenberg and Ising Δ's in three dimensions.

For Heisenberg systems with weak anisotropy, the crossover from Heisenberg to Ising behavior will be characterized by a single exponent ϕ_A which is not simply related to other exponents. This exponent will also determine the shift of T_g for small anisotropy, provided T_g of 3D Heisenberg spin glasses is nonzero. Recent torque measurements on CuMn below T_g indicate that increasing the amount of random anisotropy changes significantly the shape of the finite-field critical line.[31] If the interpretation of this change as a crossover from Heisenberg to Ising behavior is correct, then, according to the above, we expect that the nonlinear susceptibility *above* T_g should also exhibit crossover as the amount of anisotropy is changed. Our results also imply that the low-field remanent magnetization and q_{EA} (which can be measured via the ac susceptibility) should have the same critical exponent β. In addition, the critical exponent of the *macroscopic* anisotropy which was found[26] to equal $\beta_q+\beta_\Delta$ will be simply 2β below $d=6$.

So far in this paper, we have considered the effects of fluctuations on the *critical* behavior of spin-glasses. An important question is whether fluctuations can also modify the singular properties [such as the behavior of the susceptibility for small h or $q(x)$ for small x, as well as the temporal power-law decay of the correlations[23]] in the ordered SG phase well below T_g. Such modifications could be caused by *non*critical fluctuations about an ordered fixed point (or fixed line) which governs the behavior of the SG phase. This would be analogous to the fluctuation-induced divergent longitudinal susceptibility in Heisenberg ferromagnets below T_c for dimensions $2<d<4$.[32] Alternatively it is possible that the SG phase is controlled at *all* temperatures by the *critical* fixed point. This is suggested by the fact that $q(x)$ vanishes for small x which represents a lack of broken symmetry at the longest time scales.[11] In such cases, the low-T properties may be modified (perhaps already below $d=8$) by *critical* fluctuations of the type considered here. In any case, modification of the SG phase at high d may affect conclusions about the *lower* critical dimension[23] of the SG phase which are based on the assumption of mean-field properties at low temperatures.

Finally, we note that the anomalous behavior of the crossover from percolation to lattice animals[33] for $d>6$ may arise from a mechanism similar to that discussed in this paper.

We gratefully acknowledge discussions with David A. Huse concerning the effects of dangerously irrelevant variables. One of us (D.S.F.) wishes to thank the Physics Department of Bar-Ilan University for hospitality during the time when most of this work was carried out. One of us (H.S.) acknowledges the hospitality of AT&T Bell Laboratories where this work was concluded.

[1] B. Barbara, A. P. Malozemoff, and Y. Imry, Phys. Rev. Lett. **47**, 1852 (1981).

[2] R. Omari, J. J. Prejean, and J. Souletie, J. Phys. (Paris) **44**, 25 (1983).

[3] A. P. Malozemoff, S. E. Barnes, and B. Barbara, Phys. Rev. Lett. **51**, 1704 (1983).

[4] Y. Yeshurun, N. B. Salomon, K. Z. Rao, and H. S. Chen, Phys. Rev. B **24**, 1536 (1981).

[5] W.-yu Wu, G. Mozurkewich, and R. Orbach, unpublished.

[6] W. Kinzel and K. Binder, Phys. Rev. Lett. **50**, 1509 (1984); K. Binder and A. P. Young, Phys. Rev. B **29**, 2864 (1984).

[7] W. L. McMillan, J. Phys. C **17**, 3179 (1984).

[8] A. P. Young and R. N. Bhatt, unpublished.

[9] A. J. Bray and M. A. Moore, J. Phys. C **17**, L463 (1984).

[10] G. Parisi, Phys. Rev. Lett. **43**, 1754 (1979).

[11] H. Sompolinsky, Phys. Rev. Lett. **47**, 935 (1981).

[12] See, for example, S. K. Ma, in *Critical Phenomena* (Benjamin, Reading, Mass., 1976).

[13] M. Suzuki, Prog. Theor. Phys. **58**, 1151 (1977).

[14] J. R. L. deAlmeida and D. J. Thouless, J. Phys. A **11**, 983 (1978).

[15] J. H. Chen and T. C. Lubensky, Phys. Rev. B **16**, 2106 (1977); E. Pytte and J. Rudnick, Phys. Rev. B **19**, 3603 (1979).

[16] M. E. Fisher, in *Renormalization Group in Critical Phenomena and Quantum Field Theory: Proceedings of a Conference,* edited by J. D. Gunton and M. S. Green (Temple Univ. Press, Philadelphia, 1974), pp.65–72.

[17] J. E. Green, M. A. Moore, and A. J. Bray, J. Phys. C **16**, L815 (1984).

[18] We assume throughout this paper that the *structure* of the Parisi MF theory below T_g persists at least in high dimensions. In addition, we make the standard assumption that the renormalization-group recursion relations are analytic near T_g, so that we do not need to perform calculations directly in the spin-glass phase. Neither of these assumptions has been fully justified.

[19] G. Toulouse and M. Gabay, J. Phys. Lett. (Paris) **42**, L103 (1981).

[20] H. Bouchiat, J. Phys. C **16**, L145 (1983).

[21] A. J. Bray and M. A. Moore, J. Phys. C **15**, 3897 (1982).

[22] G. Kotliar and H. Sompolinsky, Phys. Rev. Lett. **53**, 1751 (1984).

[23] H. Sompolinsky and A. Zippelius, Phys. Rev. Lett. **50**, 1297 (1983); C. DeDominicis and I. Kondor, J. Phys. Lett. (Paris) **45**, L205 (1984).

[24] C. DeDominicis and I. Kondor, Phys. Rev. B **27**, 606 (1983).

[25] Y. Yeshurun and H. Sompolinsky, Phys. Rev. B **26**, 1487 (1982).

[26] H. Sompolinsky, G. Kotliar, and A. Zippelius, Phys. Rev. Lett. **52**, 392 (1984).

[27] S. E. Barnes, A. P. Malozemoff, and B. Barbara, Phys. Rev. B **30**, 2765 (1984); M. Suzuki, unpublished.

[28] P. Monod and H. Bouchiat, J. Phys. Lett. (Paris) **43**, L45 (1982).

[29] R. V. Chamberlain, M. Hardiman, L. A. Turkevich, and R. Orbach, Phys. Rev. B **25**, 6720 (1982).

[30] A. T. Ogielski and I. Morgenstern, unpublished; R. N. Bhatt and A. P. Young, unpublished.

[31] N. deCourtenay, A. Fert, and I. A. Campbell, unpublished.

[32] See, for example, G. F. Mazenko, Phys. Rev. B **14**, 3933 (1976).

[33] A. B. Harris and T. C. Lubensky, Phys. Rev. B **24**, 2656 (1981). We would like to thank Brooks Harris for a discussion of this point.