Nonlinear Stability of Drift-Tearing Modes

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The nonlinear stability of drift-tearing modes in the presence of an electron temperature inhomogeneity is investigated. It is shown that linearly growing drift-tearing modes are rendered stable at a very small island width by quasilinear thermal effects. However, both linearly and quasilinearly stabilized modes grow to large amplitude if the initial island width is larger than the linear tearing layer, demonstrating the importance of nonlinear considerations in predictions of stability.

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Magnetic reconnection in current-carrying plasmas is a process of fundamental importance both in fusion and in astrophysical research. Consequently, the linear and nonlinear behavior of tearing modes has been studied in detail. In calculations based on the equations of one-fluid magnetohydrodynamics, instability occurs when a parameter Δ', determined by the overall current profile and the mode wavelength, is positive.[1] Rutherford[2] showed that nonlinear effects become important when the magnetic island width w exceeds the rather narrow linear tearing width Δ. The growth of the islands then changes from exponential to algebraic. In high-temperature plasmas the perpendicular pressure gradient in the vicinity of the tearing layer causes the mode to propagate at the diamagnetic frequency ω_*,[3] and introduces a positive threshold Δ_c' such that if $\Delta' < \Delta_c'$ the mode is stabilized.[4–7] This threshold is a consequence of either the density[4] or the temperature[5–7] gradient. In the presence of a density gradient alone, drift-tearing modes also grow algebraically when $w > \Delta$,[8,9] while the diamagnetic propagation ceases at small w as the sound wave locally flattens this gradient.[10]

The effect of the temperature gradient on the nonlinear behavior of tearing modes is not well understood. It has been shown that small magnetic islands can locally enhance the cross-field thermal transport.[11] In this paper we are concerned with the self-consistent effects of the resulting changes in the temperature profile on the evolution of the islands. We specifically present two results based on both analytic and numerical computations. First, linearly unstable drift-tearing modes growing from small amplitude stabilize nonlinearly at $w < \Delta$ and then damp away. Quasilinear modification of the pressure profile as the island grows causes ω_* to become spatially dependent, which forces the field lines to tend to propagate at differing velocities in different parts of the tearing layer. This effect is shown below to be strongly stabilizing. Second, we demonstrate that both linearly and quasilinearly stable tearing modes, for which $\Delta' > 0$, are nonlinearly destabilized if they are initialized with $w > \Delta$. In this case parallel thermal conduction enables the temperature to equilibrate rapidly around the magnetic island, flattening the temperature gradient. All thermal stabilizing effects associated with ω_{*T} then vanish, and the island grows algebraically.

We quantify these arguments by calculating the nonlinear evolution of the tearing mode using the Braginskii[12] two-fluid equations in a low-β sheet pinch. With the electromagnetic fields given by $\mathbf{B} = B\nabla z + \nabla\psi \times \nabla z$ and $\mathbf{E} = -c^{-1}(\partial\psi/\partial t)\nabla z - \nabla\phi$, where z is the symmetry direction, the governing equations are[10]

$$\frac{1}{c}\frac{d}{dt}\psi = -\eta_\parallel J_\parallel + \frac{T}{ne}\nabla_\parallel n + \frac{\hat{\alpha}}{e}\nabla_\parallel T, \tag{1}$$

$$d(\nabla^2\phi)/dt = 4\pi(v_A^2/c^2)\nabla_\parallel J_\parallel, \tag{2}$$

$$dn/dt = e^{-1}\nabla_\parallel J_\parallel, \tag{3}$$

$$dT/dt = \tfrac{2}{3}\hat{\alpha}(T/ne)\nabla_\parallel J_\parallel + \kappa D_\parallel \nabla_\parallel^2 T. \tag{4}$$

Standard notation is used here, with $d/dt = \partial/\partial t + (c/B)\nabla z \times \nabla\phi\cdot\nabla$, $D_\parallel = T/\eta_\parallel ne^2$, $J_\parallel = -(c/4\pi)\times\nabla^2\psi$, $\nabla_\parallel = \hat{\mathbf{b}}\cdot\nabla$, $\hat{\alpha} = 1.71$, and $\kappa = 1.07$. The equations respectively represent parallel electron force balance with pressure and thermal forces, vorticity generation, continuity (allowing for parallel electron compression), and heat transport with parallel thermal conduction. We neglect parallel ion motion since the electron dynamics becomes nonlinear before the sound wave is important.

The quasilinear stabilization of linearly growing drift-tearing modes can be calculated analytically in the semicollisional limit.[13] Assuming *a priori* that $\gamma < \omega$ and that the stabilizing influence of $\tilde{\phi}$ is not important,[5–7] we linearize Eqs. (1)–(4) to obtain an equation for the perturbed flux[7,13,14]:

$$i\left(\frac{\bar{\eta}}{\Delta_D^2}\right)\tilde{\psi}_{pp} = \omega\tilde{\psi}\left(1 - \frac{\omega_{*n}(p)}{\omega} - \frac{\hat{\alpha}\omega_{*T}(p)/\omega}{1+i\kappa p^2}\right)\left(1 + ip^2 + \frac{i\bar{\alpha}p^2}{1+i\kappa p^2}\right)^{-1} \equiv \omega\tilde{\psi}\sigma(p), \tag{5}$$

where $\eta_\parallel^{-1}\sigma(p)$ is the generalized conductivity, which has been derived previously,[7,14] $\bar{\alpha} = \frac{2}{3}\hat{\alpha}^2$, $\bar{\eta} = \eta_\parallel c^2/4\pi$,

$p=x/\Delta_D$, $\Delta_D^2=\omega/k_\parallel'^2 D_\parallel$, and $k_\parallel=k_\parallel' x\equiv kx/L_s$ is the parallel wave number. In a high-temperature plasma the diamagnetic terms, $\omega_{*n}=-(ckT/neB)\langle n\rangle'$ and $\omega_{*T}=-(ck/eB)\langle T\rangle'$, on the right-hand side of Eq. (5) are individually large compared with the flux diffusion term on the left side so that small changes in these terms, which result as the growing magnetic island alters the density and temperature profiles, can seriously affect the evolution of the mode. The time dependence of ω_{*n} and ω_{*T} can be calculated from the equations for $\langle n\rangle$ and $\langle T\rangle$,

$$n^{-1}\partial\langle n\rangle/\partial t=(1/ne)\langle\tilde{\nabla}_\parallel\tilde{J}_\parallel\rangle,$$

$$T^{-1}\partial\langle T\rangle/\partial t=\tfrac{2}{3}(\hat{\alpha}/ne)\langle\tilde{\nabla}_\parallel\tilde{J}_\parallel\rangle+\kappa D_\parallel T^{-1}\langle\tilde{\nabla}_\parallel(\nabla_\parallel T)^\sim\rangle, \quad (6)$$

in which the angle brackets represent an average over y, the tilde denotes a fluctuating quantity with the dependence $\exp[iky-i\int^t\omega(t')dt']$, and it is understood that the ω_* terms take their linear values at $t=0$. We first evaluate the evolution of the temperature. With $\tilde{\nabla}_\parallel=i(k\tilde{\psi}/B)(\partial/\partial x)$, $\tilde{J}_\parallel=i\omega\sigma(p)\tilde{\psi}/\eta_\parallel c$, and linearization of Eq. (4) to obtain $\tilde{T}$, Eq. (6) becomes

$$\frac{1}{T}\frac{\partial}{\partial t}\langle T\rangle=2\frac{D_\parallel}{L_n\Delta_D}\left|\frac{k\tilde{\psi}}{B}\right|^2\mathrm{Re}\frac{\partial}{\partial p}\{[\tfrac{2}{3}\hat{\alpha}\hat{\omega}\sigma(p)-\kappa\eta_e](1+i\kappa p^2)^{-1}\}, \quad (7)$$

where $\hat{\omega}=\omega/\omega_*$, $\omega_*=ckT/eBL_n$, $\eta_e=L_n/L_T$, and L_n and L_T are the scale lengths of density and temperature at $t=0$. The spatial gradients of $\langle n\rangle$ and $\langle T\rangle$ in Eq. (7) have been replaced by their initial values. The time dependence of $\omega_{*T}(p)$ is then given by

$$\omega_{*T}(p)=\eta_e\omega_*\left\{1+\delta_*\mathrm{Re}\frac{d^2}{dp^2}\left[\left(\frac{2}{3}\frac{\hat{\alpha}}{\eta_e}\hat{\omega}\sigma(p)-\kappa\right)(1+i\kappa p^2)^{-1}\right]\right\}, \quad (8)$$

$$\partial\delta_*/\partial t=\omega(w/2\Delta_D)^4/2,$$

where $w=(8|\tilde{\psi}|L_s/B)^{1/2}$ is the island width. Note that in the expression for $\omega_{*T}(p,t)$ in Eq. (8) the dependences on space and time are separable, a result which will enable us to solve analytically for the complete quasilinear behavior of the mode. After calculating a similar expression for $\omega_{*n}(p,t)$ and integrating Eq. (5) across the tearing layer ($p\sim 1$), we find

$$i\bar{\eta}\Delta'/\Delta_D=\omega_* e^{-i\pi/4}[I_1(\hat{\omega}-1)-I_2\hat{\alpha}\eta_e+I_3\eta_e\delta_*],$$

where $\Delta'=(\tilde{\psi}\Delta_D)^{-1}\int_{-\infty}^{\infty}dp\,\tilde{\psi}_{pp}$ is the usual tearing-mode stability parameter,[1] $I_1=2.77$ and $I_2=1.36$ are obtained from linear theory, and

$$I_3=e^{i\pi/4}\int_{-\infty}^{\infty}dp\left[\{1+ip^2[1+\bar{\alpha}(1+i\kappa p^2)^{-1}]\}^{-1}\times\left(\mathrm{Re}\frac{d^2}{dp^2}[\eta_e^{-1}\hat{\omega}\sigma(p)]+\frac{1}{1+i\kappa p^2}\mathrm{Re}\frac{d^2}{dp^2}\{(1+i\kappa p^2)^{-1}[\bar{\alpha}\eta_e^{-1}\hat{\omega}\sigma(p)-\kappa\hat{\alpha}]\}\right)\right]$$

$$=-4.91+1.49i.$$

The linear mode frequency has been used in evaluation of I_3.[7,14] The growth rate is then given by

$$\gamma=\gamma_L-0.538\eta_e\omega_*\delta_*,$$

where $\gamma_L=\bar{\eta}\Delta'/\sqrt{2}I_1\Delta_D$ is the linear growth rate. Since $\delta_*>0$, the quasilinear modification of the pressure profile is clearly stabilizing. Differentiating γ with respect to time, we find

$$\partial\gamma/\partial t=-A\omega_*^2(w/\Delta_D)^4, \quad (9)$$

where $A=0.0186\eta_e(1+0.84\eta_e)$. The quasilinear description is completed with the equation for island growth,

$$\partial w/\partial t=\gamma w/2. \quad (10)$$

The solution of Eqs. (9) and (10) is readily calculable and is given by

$$\gamma=\gamma_L\tanh[\gamma_L(t_s-t)], \quad (11)$$

$$w/\Delta_D=(\gamma_L/\omega_* A^{1/2})^{1/2}\mathrm{sech}^{1/2}[\gamma_L(t_s-t)], \quad (12)$$

with $t_s=\gamma_L^{-1}\ln(2\gamma_L\Delta_D^2/\omega_* w_0^2 A^{1/2})$ and w_0 the initial island width. Note that the maximum island size occurs at $t=t_s$ and is approximately

$$w_{\max}\approx(\gamma_L/\omega_*)^{1/2}\Delta_D<\Delta_D, \quad (13)$$

where the inequality is valid because a drift-tearing mode has by definition $\gamma_L<\omega_*$. As $t\to\infty$, the damping rate becomes $\gamma=-\gamma_L$ so that $w\to 0$.

In this way small-amplitude linearly growing drift-tearing modes quasilinearly stabilize at small island widths. The quasilinear approach is justified since $w < \Delta_D$ for all t. For $w \sim \Delta_D$ nonlinear and linear terms become comparable and the quasilinear theory breaks down.[15] Finally, we must emphasize that both the temperature gradient and thermal conduction are required for quasilinear stabilization. This is evident from Eq. (9) as $A \propto \eta_e$. For $\eta_e = 0$, there is a large cancellation between ω and ω_{*n} in Eq. (5), so that $\sigma(p) \sim 1 - \omega_{*n}/\omega \sim \gamma/\omega_{*n}$. As a consequence, the quasilinear modifications of $\langle n \rangle$ and $\langle T \rangle$ are greatly reduced [see Eq. (7)]. Similarly, if $\eta_e \neq 0$ but $\kappa = 0$, the temperature behaves like the density, and the quasilinear effects are small.

In support of our analysis we have undertaken a numerical solution of Eqs. (1)–(4). We use a finite-difference approximation in the x direction, expand y dependences into harmonics, and restrict the calculation to a region small compared to L_n in a way described previously.[10] This nonlinear, initial-value prescription has been checked in the linear regime for a range of plasma parameters by the ensuring of agreement of the mode frequency and growth rate with the results of a shooting code[7] which solves the linearizations of Eqs. (1)–(4). The results of a representative case of quasilinear stabilization are presented in Fig. 1. The initially very small island width is seen to rise and fall with time as the growth rate drops through zero (solid lines). The curves described by Eqs. (11) and (12) are shown for comparison (dashed lines), differences being due to the asymptotic nature of the quasilinear theory. The inclusion of higher harmonics has been found to produce only slight differences in the maximum island width and its timing; the qualitative behavior remains unchanged.

The nonlinear behavior of drift-tearing modes changes dramatically, however, when the initial island width is as large as Δ_D. We evolve Eqs. (1)–(4) with initial conditions in which $J = J(\psi)$ and $T = T(\psi)$. The results are best understood in the context of the parameter space introduced in Ref. 7, in which the parameters $\hat{\beta} \equiv \beta L_s^2/L_n^2$ and $C \equiv 0.51(\nu_e/\omega_*)(m/M)L_s^2/L_n^2$ were shown to determine the properties of the linear mode. For a given $C < 1$ (the semicollisional regime), linear tearing modes are nearly purely growing if $\hat{\beta}/\Delta'\rho_s < C^{1/2}$, are drift dominated and unstable if $C^{1/2} < \hat{\beta}/\Delta'\rho_s < 1$, and are stabilized by the influence of $\tilde{\phi}$ if $\hat{\beta}/\Delta'\rho_s > 1$. We have numerically calculated the nonlinear evolution of the islands with initial widths ranging from a fraction of to several times Δ_D for values of $\hat{\beta}$ above and below the linear stability boundary. The nonlinear stability of the system for a given $\hat{\beta}$ and w_0 is indicated in Fig. 2. For initial conditions labeled by the filled (open) circles the islands grow to large amplitude (damp away). Note that an increase in $\hat{\beta}$ reflects an increase of ω_*. The dashed curve shown separates the nonlinear stable and unstable regions. The point on the vertical axis touched by this curve is the minimum value of $\hat{\beta}/\Delta'\rho_s$ for which the linear mode is stabilized by quasilinear effects, and the point marked L is the linear stability boundary. It is clear that an initial island width of just a few Δ_D is sufficient to destabilize both linearly and

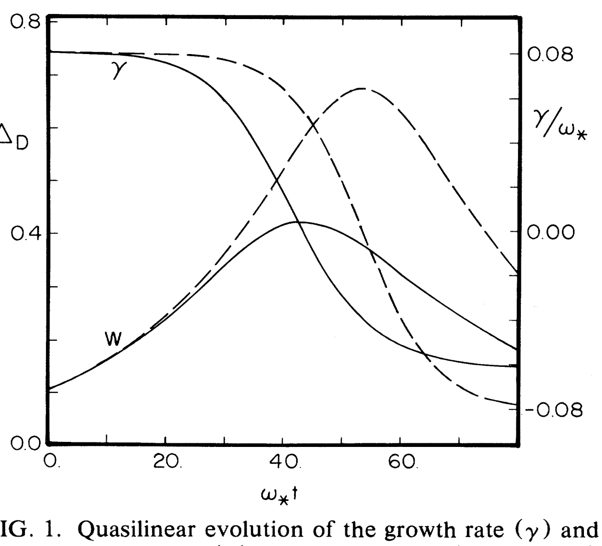

FIG. 1. Quasilinear evolution of the growth rate (γ) and magnetic island width (w), from numerical (solid curves) and analytical (dashed curves) calculations. The numerical linear growth rate was used for γ_L in Eqs. (11) and (12).

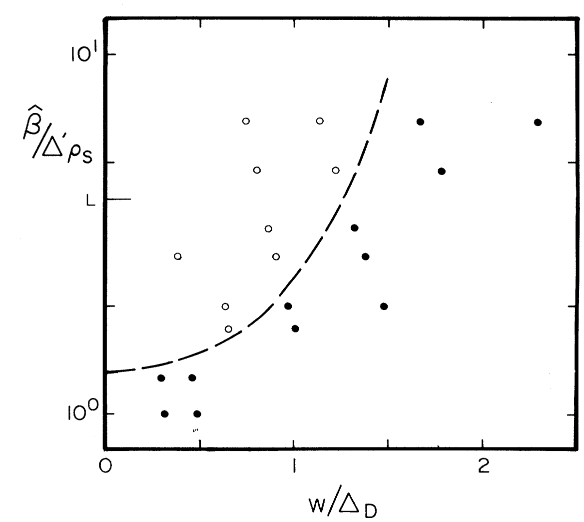

FIG. 2. Parameter plot indicating nonlinear stability of the system to drift-tearing modes. Filled circles represent cases in which the mode grows to large amplitude; open circles signify modes which damp away. The point L denotes the linear stability boundary, while the transition into the drift regime is below the frame at $\hat{\beta}/\Delta'\rho_s = 0.33$.

quasilinearly stabilized modes. These results are not inconsistent with previous reports that large islands are not affected by thermal forces.[16] The physically different case of $\hat{\beta} > 1$, in which linear tearing modes are stabilized by diamagnetic shielding,[7] is presently under investigation.

The significance of our results is that the conclusions of linear drift-tearing mode stability theory are diminished in relevance as far as operations on tokamaks are concerned. This is because the initial perturbation required for continued growth is so small: Δ_D is of order 1 mm on the Princeton Large Torus and scales as $T^{-3/4}$. Perturbations due to magnetic field errors, ripple, or residual islands are likely to be larger than this. Moreover, the smallest $m=2$ oscillations presently measurable in tokamaks in fact are already in the nonlinearly unstable regime. The question of stability is still open for the collisionless regime, as it is for the case that $\hat{\beta} > 1$. Nevertheless, a confident stability prediction in either of these cases awaits a complete nonlinear treatment. Our results demonstrate the importance of nonlinear calculations in general in assessing the overall stability of a current-carrying plasma.

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