Production of Squeezed States in the Interaction between Electromagnetic Radiation and an Electron Gas

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(Received 26 October 1984)

We show that squeezed states can be generated by the interaction of coherent electromagnetic radiation with a plasma. Inside the plasma the electromagnetic field is represented by quasiphotons. The conditions for obtaining squeezed states in this system are related to dispersion relations of the quasiphotons in the plasma.

PACS numbers: 42.50.+q, 52.25.Rv

There is today much interest in generating squeezed states of the electromagnetic (e.m.) field.[1-10] These states are characterized by the fact that the variance of one of the two quadrature components of the e.m. field is smaller than the minimal product of the two uncertainties. The squeezed states are interesting also because they exhibit nonclassical features. In spite of the recent theoretical progress made in this field, so far, these states have not been observed.

In the present work we show that the e.m. field in the plasma may be represented by quasiphotons. [The adjective "quasi" indicates that the dispersion relation $\omega(k)$ is different from that of the free photon $\omega = ck$.] We discuss the conditions for obtaining squeezed states by the interaction of the radiation with the plasma.

The elementary dispersion relations of electromagnetic waves in plasma are derived usually by using the classical Maxwell equations, in which one takes into account currents due to first-order charged-particle motions.[11] The quantum mechanical treatment of the coupling between the electrons and the e.m. field gives the same dispersion relations.[12] Usually, one is interested in the dispersion relations of e.m. waves in plasma, disregarding the possible influence of the interaction with the plasma on the photon statistics of the radiation field. The properties of squeezed states cannot be calculated by classical methods, as these states do not possess nonsingular representation in terms of the Glauber-Sudarshan P distribution.[1,2] The quantum approach developed in the present article is therefore indispensible for the analysis of squeezed states generated by the interaction of an e.m. wave and a plasma, although it gives the same dispersion relations as those derived by classical methods.

Usually there is "a competition between the squeezing produced by the nonlinear interaction and the degradation of the squeezing by the damping."[2] In most systems[1-10] vacuum fluctuations due to spontaneous emission and damping mechanism "will tend to equalize the variance in the two quadratures and hence destroy the squeezing."[2] We believe that in the present system these effects are minimal. This belief is based on the fact that in our derivations the appearance of squeezing is intimately tied with the derivation of the well-known dispersion relation of e.m. waves in plasma. As this relation is well established[11] we expect the same validity to hold also for squeezing.

A coherent state $|\alpha\rangle$ is generated by the action of the displacement operator $D_a(\alpha)$ on the vacuum: $|\alpha_a\rangle = D_a(\alpha)|0_a\rangle$ where $D_a(\alpha) = \exp[\alpha a^\dagger - \alpha^* a]$. The index a refers here to a certain single mode of the radiation with wave vector $\mathbf{k}$ and frequency $\omega = kc$. A squeezed state $S_a(\alpha,z)$ may be generated by first acting with the unitary squeeze operator $S_a(z)$ on the vacuum followed by the displacement operator[2,8]

$$|S_a(\alpha,z)\rangle = D_a(\alpha)S_a(z)|0_a\rangle, \tag{1}$$

where

$$S_a(z) = \exp[\tfrac{1}{2}za^2 - \tfrac{1}{2}z^* a^{\dagger 2}]$$

and $z = re^{-i\theta}$. The squeezed states are the eigenstates of the operator b:

$$b|S_a(\alpha,z)\rangle = \beta|S_a(\alpha,z)\rangle, \tag{2}$$

$$b = S_a(z)aS_a^{-1}(z) = aC_r + a^\dagger e^{i\theta}S_r, \tag{3}$$

$$C_r = \cosh r, \quad S_r = \sinh r. \tag{4}$$

Let us consider a physical system with the general quadratic Hamiltonian[6]

$$H = \hbar(\omega a^\dagger a + f_1^* a + f_1 a^\dagger + f_2^* a^2 + f_2 a^{\dagger 2}), \tag{5}$$

where the c-numbers ω, f_1, and f_2 may be time dependent. Using the Bogoliubov transformation[13] represented by (3) with the values of C_r and S_r fixed by

$$\begin{aligned} &C_r S_r e^{i\theta} = f_2/\omega_0, \quad \omega_0^2 = \omega^2 - 4|f_2|^2, \\ &C_r = [(\omega+\omega_0)/2\omega_0]^{1/2}, \\ &S_r = [(\omega-\omega_0)/2\omega_0]^{1/2}, \end{aligned} \tag{6}$$

we get

$$H = \hbar\omega_0[b^\dagger b + \beta^* b + \beta b^\dagger - S_r^2], \tag{7}$$

where

$$\omega_0\beta = f_1 C_r - f_1^* e^{i\theta} S_r. \tag{8}$$

The ground state of the Hamiltonian (7) is equivalent to the squeezed state. $b^\dagger$ is interpreted as the "quasiphoton" creation operator.

We would like now to show that the system of an electron gas interacting with a monochromatic coherent radiation field is a good candidate for producing squeezed states. We assume a plasma in which the direct interactions between electrons may be ignored (the Debye screening length is small relative to the average distance between the electrons). We assume low velocities for the electrons and neglect the effects of electron-ion and electron-atom collisions. We call the electrons of such a plasma "free" electrons.

The Hamiltonian of N "free" electrons (with coordinates $\mathbf{r}_i$ and momenta $\mathbf{p}_i$) interacting with a coherent e.m. field may be written as

$$\begin{aligned} H &= \frac{1}{2m}\sum_{i=1}^{N}[\mathbf{p}_i - \frac{e}{c}\mathbf{A}(\mathbf{r}_i)]^2 + H_{0R} \\ &= \frac{1}{2m}\sum_{i=1}^{N}p_i^2 + H_{0R} - \frac{e}{m}\sum_{i=1}^{N}\mathbf{p}_i\cdot\sum_{\mathbf{k}}\left(\frac{2\pi\hbar}{\omega_k V}\right)^{1/2}\hat{\mathbf{e}}_{\mathbf{k}}[(a_{\mathbf{k}}+f_{\mathbf{k}})e^{i\mathbf{k}\cdot\mathbf{r}_i} + (a_{\mathbf{k}}^\dagger + f_{\mathbf{k}}^*)e^{-i\mathbf{k}\cdot\mathbf{r}_i}] \\ &\quad + \frac{e^2}{2m}\frac{2\pi\hbar}{V}\sum_{\mathbf{k}\mathbf{k}'}\left(\frac{1}{\omega_k'\omega_k}\right)^{1/2}(\hat{\mathbf{e}}_{\mathbf{k}}\cdot\hat{\mathbf{e}}_{\mathbf{k}'})\{[\rho_{-(\mathbf{k}+\mathbf{k}')}(a_{\mathbf{k}}+f_{\mathbf{k}})(a_{\mathbf{k}'}+f_{\mathbf{k}'}) + \text{H.c.}] \\ &\qquad + \rho_{-(\mathbf{k}-\mathbf{k}')}(a_{\mathbf{k}}+f_{\mathbf{k}})(a_{\mathbf{k}'}^\dagger + f_{\mathbf{k}'}^*) \\ &\qquad + \rho_{\mathbf{k}-\mathbf{k}'}(a_{\mathbf{k}}^\dagger + f_{\mathbf{k}}^*)(a_{\mathbf{k}'}+f_{\mathbf{k}'})\}, \end{aligned} \tag{9}$$

$$H_{0R} = \hbar\sum_{\mathbf{k}}\omega_k(a_{\mathbf{k}}^\dagger + f_{\mathbf{k}}^*)(a_{\mathbf{k}}+f_{\mathbf{k}}), \quad \hat{\mathbf{e}}_{\mathbf{k}}\cdot\mathbf{k} = 0. \tag{10}$$

Here $\rho_{\mathbf{k}} = \sum_{i=1}^{N}\exp[-i\mathbf{k}\cdot\mathbf{r}_i]$, $a_{\mathbf{k}}, a_{-\mathbf{k}}^\dagger$ are the annihilation and creation operators for the mode $\mathbf{k}$, and $f_{\mathbf{k}}$ represents the strength of the classical driving field, at this mode. Here we used a short notation where the summation over $\mathbf{k}$ includes for each wave vector $\mathbf{k}$ two linear polarizations $\hat{\mathbf{e}}_{\mathbf{k}\lambda}$ ($\lambda = 1, 2$). The radiation field in the Hamiltonian (9) represents driven oscillators for all modes for which $f_{\mathbf{k}} \neq 0$.

Assuming that the location of the electrons is random and that the dimensions of the system are large relative to one wavelength, we can neglect the terms which are linear in $\mathbf{p}_i$ and use in the quadratic terms $\sum_{i=1}^{N}\mathbf{A}^2(\mathbf{r}_i)$ the following relation:

$$\rho_{\mathbf{k}} \approx \delta_{\mathbf{k},0}N. \tag{11}$$

Neglecting all terms which do not include the e.m. field operators $a_{\mathbf{k}'}, a_{\mathbf{k}}$, we get for the Hamiltonian of the radiation field

$$H_R = \hbar\sum_{\mathbf{k},\lambda}\Omega_k a_{\mathbf{k},\lambda}^\dagger a_{\mathbf{k},\lambda} + \hbar\sum_{\mathbf{k},\lambda}\left[\frac{\omega_p^2}{4\omega_k}(a_{\mathbf{k},\lambda}^\dagger a_{-\mathbf{k},\lambda}^\dagger + a_{\mathbf{k},\lambda}a_{-\mathbf{k},\lambda})\right] + \hbar\sum_{\mathbf{k},\lambda}[f_{1,\mathbf{k},\lambda}^* a_{\mathbf{k},\lambda} + f_{1\mathbf{k},\lambda}a_{\mathbf{k},\lambda}^\dagger]. \tag{12}$$

Here[14]

$$\Omega_k = \omega_k + (2\pi e^2 N/m\omega_k V) = \omega_k(1 + \omega_p^2/2\omega_k^2)$$

where $\omega_p = (4\pi e^2 N/mV)^{1/2}$ is the plasma frequency of the electron gas[11,12] and $f_{1,\mathbf{k},\lambda} = \Omega_k f_{\mathbf{k},\lambda} + (\omega_p^2/2\omega_k)f_{-\mathbf{k},\lambda}^*$. According to Eq. (12) the mode $\mathbf{k}, \lambda$ is coupled only to the mode $-\mathbf{k}, \lambda$ so that the Hamiltonian of radiation can be separated into the summation of Hamiltonians for pairs of modes. For a certain pair of modes $\mathbf{k}, \lambda$ and $-\mathbf{k}, \lambda$ we introduce new boson operators

$$a_+ = (a_{\mathbf{k},\lambda} + a_{-\mathbf{k},\lambda})/\sqrt{2}; \quad a_- = i(a_{\mathbf{k},\lambda} - a_{-\mathbf{k},\lambda})/\sqrt{2}. \tag{13}$$

The Hamiltonian of radiation for the pair of modes $\mathbf{k}, \lambda$ and $-\mathbf{k}, \lambda$ is given by

$$\begin{aligned} H_{p,\mathbf{k},\lambda} &= \hbar\Omega[a_+^\dagger a_+ + a_-^\dagger a_-] + \frac{\hbar\omega_p^2}{4\omega}[a_+^\dagger a_+^\dagger + a_-^\dagger a_-^\dagger + a_+ a_+ + a_- a_-] \\ &\quad + \hbar[a_+ f_+^* + a_+^\dagger f_+ + a_- f_-^* + a_-^\dagger f_-], \end{aligned} \tag{14}$$

where

$$f_+ = (f_{1,\mathbf{k},\lambda} + f_{i,-\mathbf{k},\lambda})/\sqrt{2}; \quad f_- = i(f_{i,\mathbf{k},\lambda} - f_{i,-\mathbf{k},\lambda})/\sqrt{2}. \tag{15}$$

Equation (14) represents two uncoupled modes of standing waves with separated Hamiltonians given by

$$H_+ = \hbar\Omega a_+^\dagger a_+ + \hbar\frac{\omega_p^2}{4\omega}[a_+^\dagger a_+^\dagger + a_+ a_+] + \hbar[a_+ f_+^* + a_+^\dagger f_+], \tag{16}$$

and a similar equation for H_-. Let us consider a driven oscillator of the radiation field at one standing mode as represented by H_+. We ignore all other modes of the radiation field by assuming[15] $f_{\mathbf{k},\lambda} = f_{-\mathbf{k},\lambda}$ (so that $f_- = 0$) and $f_{\mathbf{k}'} = 0$ for $\mathbf{k}' \neq \pm\mathbf{k}$. Since the Hamiltonian H_+ [Eq. (16)] with the boson operators $a_+, a_+^\dagger$ is of the same form as Eq. (5) with the boson operators $a, a^\dagger$, we transform to operators $b_+, b_+^\dagger$ according to Eqs. (3) and (4) and obtain [see Eqs. (6) and (7)]

$$H_+ = \hbar\nu(b_+^\dagger b_+ + b_+\beta_+^* + b_+^\dagger\beta_+ - S_r^2). \tag{17}$$

In our case

$$\nu^2 = \Omega^2 - 4|f_2|^2 = \omega^2\left(1 + \frac{\omega_p^2}{2\omega^2}\right)^2 - \frac{\omega_p^4}{4\omega^2}$$

$$= \omega_p^2 + \omega^2.$$

Since $\omega = kc$, we can identify $\nu = (\omega_p^2 + k^2c^2)^{1/2}$ with the well-known dispersion relation for e.m. waves in plasma.

The eigenstates of the Hamiltonian $\hbar\Omega a_+^\dagger a_+$ are exactly the eigenstates of the Hamiltonian $\hbar\omega a_+^\dagger a_+$. Therefore the change of frequency from ω to Ω does not cause any changes in the statistics of the radiation field. However, the final result, i.e., the change of the dispersion relation of $\nu = (\omega_p^2 + k^2c^2)^{1/2}$, is well established experimentally and therefore proves by our analysis the reality of the quasiphotons. We also conclude that similar changes in the dispersion relation should happen in other devices for producing squeezed states by an effective quadratic Hamiltonian. The change in the dispersion relation inside the interaction region would be an indication of the formation of quasiphotons and squeezed states.

Let us study the properties of our squeezed states by examining the explicit expression for β_+ and b_+. b_+ is defined by (3) where in our case

$$\tanh(2r) = \omega_p^2/(\nu^2 + k^2c^2), \quad \theta = 0. \tag{18}$$

ν is the frequency of the e.m. radiation while k is the wave number in the plasma. The maximal value of $\tanh(2r)$ is 1 when $kc \ll \omega_p$, so that by changing the density of electrons in the plasma we cover all possible Bogoliubov transformations for real values of z. The ground state of the Hamiltonian (17) is the squeezed state which, because z is real, is a minimum-uncertainty state both in operators $a_+, a_+^\dagger$ and $b_+, b_+^\dagger$.

The value of β_+ is given by [see Eq. (8)]

$$\nu\beta_+ = f_+ C_r - f_+^* S_r;$$

$$f_+ = \sqrt{2}[\Omega_k f_{\mathbf{k},\lambda} + (\omega_p^2/2\omega_k) f_{\mathbf{k},\lambda}^*]. \tag{19}$$

We can use Eqs. (7) and (8) for the standing mode of radiation when we replace a, b, and β, respectively, by a_+, b_+, and β_+.[16]

Finally, we would like to point out that we have obtained the squeezed states for a standing mode of radiation, where in our case the quadratic terms of the Hamiltonian do not change the momentum of the system. The same situation can occur in a two-photon laser where two photons of the same frequency with wave vectors $\mathbf{k}$ and $-\mathbf{k}$ can be absorbed in a transition which is of second order in $\mathbf{p}\cdot\mathbf{A}$.[6]

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[15]This simplifies the following analysis but is not a necessary condition for squeezing. An alternative approach to the present treatment would be to diagonalize the first two parts of Eq. (16) directly by the Bogoliubov transformation: $b_{\mathbf{k}} = C_r a_{\mathbf{k}} + a_{-\mathbf{k}}^\dagger e^{i\theta} S_r$.

[16]The ground state of H_{0R} [Eq. (14)] for a certain mode $\mathbf{k}$ describes a coherent state of $a_{\mathbf{k}}$, which may be produced by a one-mode laser. The ground state of H_+ [Eq. (20)] describes a coherent state of b_+ (with the same $\mathbf{k}$) which may be produced by adding the plasma to the one-mode laser, and is a squeezed state in terms of a_+.