Ultraviolet Divergences in $1/N$ Expansions of Quantum Field Theories

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For asymptotically free theories, ultraviolet divergences computed in the $1/N$ expansion with dimensional regularization reduce to simple poles plus powers of $\ln\epsilon$ or finite terms. The theories are effectively superrenormalizable since all divergences are determined by the one- and two-loop perturbative renormalization-group functions.

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Perturbative renormalizability has served as a powerful tool in the selection of physically admissible models, most notably in the development of the non-Abelian gauge theory of electroweak and strong interactions. It may be that only asymptotically free theories are actually renormalizable. In such theories the high-momentum behavior of irreducible vertex functions calculated to all orders differs only logarithmically from the growth predicted by naive power counting. With infrared free theories, however, the vertex functions may grow by extra powers of the large momentum because the anomalous dimensions do not vanish in the ultraviolet limit. This means that the divergence of the full theory may be worse than those encountered order by order in perturbation theory and that the renormalization scheme may break down when an infinite set of diagrams is summed.[1]

In this paper we report some results on nonperturbative renormalization of asymptotically free theories which support and extend these ideas. More detailed accounts of the calculations and of their extension to infrared free theories are presented elsewhere.[2,3] We consider field-theory models which admit expansions in powers of $1/N$ where N is the number of internal components of a field.[4] Previous investigations of these expansions have concentrated generally on calculation of their behavior to leading order in $1/N$ or on exact solution of the S matrices and spectra. Though there have been general discussions of the renormalizability of Green's functions in $1/N$ expansions of various theories, very few explicit calculations of the ultraviolet divergences beyond the leading order have appeared.

With use of dimensional regularization to carry out such calculations to the next to leading order, we have discovered new results. In perturbation theory the singularities associated with such a regularization are poles in ϵ, where ϵ is the number of extra dimensions of space-time. For asymptotically free models we find that beyond the leading order in $1/N$ where simple poles in ϵ occur, the ultraviolet divergences consist only of additional simple poles plus powers of $\ln\epsilon$.

These results are supported by calculations based on the renormalization-group functions determined from perturbation theory. The renormalization-group analysis leads to the further conclusion that the ultraviolet divergences of the $1/N$ expansion for asymptotically free theories are determined completely by the one- and two-loop divergences of the perturbative expansions. Thus, these theories are effectively superrenormalizable in the $1/N$ expansion. The results leading to these conclusions are summarized below.

The asymptotically free models we have considered explicitly are the nonlinear sigma model (NLSM)[5] and the Gross-Neveu model (GNM)[6] near $d=2$. The Lagrangian density for the former can be written as

$$\mathscr{L}=\tfrac{1}{2}(\partial_\mu\eta^j\partial_\mu\eta^j)-\tfrac{1}{2}\lambda(\eta^j\eta^j-N/g^2), \tag{1}$$

where $j=1,\ldots,N$ labels the components of the scalar field η, and λ is a Lagrange multiplier field which implements the nonlinear constraint in the parentheses. In the $1/N$ expansion a mass is dynamically generated for the η quanta. Green's functions involving external η particles require wave-function and coupling-constant renormalizations with the corresponding constants Z_η and Z_α ($\alpha=g^2/2\pi$). Both can be determined by calculating the self-energy of the η field. To $O(1/N)$ we find

$$Z_\eta=1+(1/N)\ln\epsilon,$$

and

$$Z_\alpha^{-1}=1+\alpha[(1-2/N)/\epsilon-(1/N)\ln\epsilon]. \tag{2}$$

The perturbative renormalization-group functions for this model are[7]

$$\beta(\alpha)=-\left(1-\frac{2}{N}\right)\alpha^2\left[1+\frac{\alpha}{N}+O\left(\frac{\alpha^2}{N}\right)\right], \tag{3}$$

and

$$\gamma_\eta = \alpha/2N + O(\alpha^2/N). \tag{4}$$

The corresponding renormalization constants are determined from

$$\ln Z_\alpha^{-1} = \int_0^\alpha \frac{\beta(x)/x^2}{[-\epsilon+\beta(x)/x]}dx, \tag{5}$$

and

$$\ln Z_\eta = 2\int_0^\alpha \frac{\gamma_\eta(x)/x^2}{[-\epsilon+\beta(x)/x]}dx, \tag{6}$$

where $d = 2-\epsilon$. (Note that for $\epsilon > 0$ for $d < 2$.)

We integrate to find Z_α using the two-loop β function. Actually, it is convenient to use the modified form

$$\beta(\alpha) = -a\alpha/(1-b\alpha), \qquad a = 1-2/N, \quad b = 1/N, \tag{7}$$

which differs from the two-loop answer only in higher orders. We show later that the divergences of Z_α are insensitive to such higher-order contributions. Substituting in Eq. (5) gives

$$\ln Z_\alpha^{-1} = [1-b\epsilon/a]^{-1}\ln[1+(a-b\epsilon)\alpha/\epsilon]. \tag{8}$$

Exanding in powers of α gives

$$\ln Z_\alpha^{-1} = \frac{a\alpha}{\epsilon}\sum_{k=0}\frac{(-1)^k}{(k+1)}\left(\frac{a-b\epsilon}{\epsilon}\alpha\right)^k, \tag{9}$$

which has the expected structure as a sum of poles. However, an expansion in powers of $b \sim O(1/N)$ gives

$$Z_\alpha^{-1} = 1 + \frac{a\alpha}{\epsilon} - b\alpha\ln\epsilon + \text{(finite terms)}, \tag{10}$$

which agrees exactly with our explicit calculation. With the one-loop anomalous dimension of Eq. (4) and the β function of Eq. (7), the wave-function renormalization constant is given by Eq. (6) as

$$\ln Z_\eta = \frac{2c}{a-b\epsilon}\left[\frac{a}{a-b\epsilon}\ln\left(1+\frac{a-b\epsilon}{\epsilon}\alpha\right)-b\alpha\right], \tag{11}$$

$$c = -1/2N.$$

Expanding in powers of $1/N$ and letting $\epsilon \to 0$ gives

$$\ln Z_\eta = \frac{1}{N-2}\ln\epsilon + \text{(finite terms)}, \tag{12}$$

which agrees with our explicit calculation of Z_η to $O(1/N)$.

Furthermore, the higher-order perturbative contributions to β and γ give only terms which do not diverge as $\epsilon \to 0$. These extra terms can be scaled away by finite renormalizations and the forms given in Eqs. (10) and (12) with the finite pieces dropped are sufficient to renormalize the theory in the $1/N$ expansion. To see this explicitly, write

$$\beta(\alpha) = a\alpha^2[1+b\alpha+\alpha^2 h(\alpha)], \tag{13}$$

and

$$\gamma(\alpha) = c\alpha + \alpha^2 g(\alpha), \tag{14}$$

where $h(\alpha)$ and $g(\alpha)$ are $O(1/N)$ and finite at $\alpha = 0$. Using the superscript (2) to denote two-loop quantities, we find

$$\ln Z_\alpha^{(2)} - \ln Z_\alpha = \frac{\epsilon}{a}\int_0^\alpha \frac{h(x)}{(1+bx)[1+bx+x^2h(x)]}dx + O(\epsilon^2\ln\epsilon), \tag{15}$$

and

$$\ln Z_\eta^{(2)} - \ln Z_\eta = \int_0^\alpha \frac{ch(x)+ag(x)(1+bx)}{a^2(1+bx)[1+bx+x^2h(x)]}dx + O(\epsilon\ln\epsilon). \tag{16}$$

Similar results appear in the GNM at $d = 2$. The Lagrangian density is

$$\mathscr{L} = \overline{\Psi}_j i\partial\!\!\!/\Psi_j + (N/2g^2)(\overline{\Psi}_j\Psi_j)^2. \tag{17}$$

The theory requires a wave-function and a coupling-constant renormalization. Defining $\alpha = g^2/\pi$ in this case, calculation of the self-energy to $O(1/N)$ gives

$$Z_\alpha^{-1} = 1 + \alpha[(1-1/N)\epsilon^{-1} + (1/2N)\ln\epsilon], \tag{18}$$

and

$$Z_\Psi = 1 + \text{(finite terms)}. \tag{19}$$

To the required order the perturbative renormalization-group functions are

$$\beta(\alpha) = -(1-1/N)\alpha^2[1-\alpha/2(N-1)], \tag{20}$$

and

$$\gamma_\Psi(\alpha) = \alpha^2/8N. \tag{21}$$

The charge-renormalization constant with the minimally required divergent structure may be found as for the NLSM with the result

$$Z_\alpha^{-1} = 1 + (1-1/N)\alpha\epsilon^{-1} + [\alpha/2(N-1)]\ln\epsilon \tag{22}$$

while Z_Ψ is finite. These results agree with our explicit calculations to $O(1/N)$.

Though the predictions of the ultraviolet singularities of the $1/N$ expansions can be deduced directly from the renormalization constants determined from low-order perturbation theory, we would not have been confident about accepting these results without the support of explicit calculations. Each order of the $1/N$ expansion includes an infinite subset of the perturbative Feynman graphs. With the evidence of these model calculations to validate the procedure, however, we can apply the renormalization-group method to an important case were explicit $1/N$ calculations have not yet proved feasible, namely, non-Abelian gauge theories.

For an SU(N) theory with N_f flavors of fermions in the fundamental representation and with the gauge coupling constant written as $g^2=\alpha/N$ the two-loop β function is[8]

$$\beta(\alpha) = -B_0\alpha^2 + B_1\alpha^3, \tag{23}$$

with

$$B_0 = (8\pi^2)^{-1}[11/3 - 2N_f/3N],$$

and

$$B_1 = \frac{1}{(16\pi^2)^2}\left[-\frac{34}{3} + \frac{10N_f}{3N} + \frac{(N^2-1)}{N^2}N_f^2\right].$$

This β function can be written in the form of Eq. (7) with $a=B_0$ and $b=-B^0/B_1$. In this case $b\sim O(1)$, but the singular part of Z_α^{-1} is still given by Eq. (10). The $\ln\epsilon$ singularity enters at the zeroth order of $1/N$ rather than at the first order as in the two-dimensional models. This is not surprising since the lowest-order contributions for the NLSM and the GNM are sums of an infinite set of bubble graphs, which require only single-pole renormalizations, while the lowest-order terms for gauge theories consist of all planar graphs which have more complicated divergences.

The question remains as to which divergent structure reflects the true behavior of the theory, perturbation theory or the $1/N$ expansion. For the NLSM and GNM, the exact S matrices are known and their only natural expansion parameter is $1/N$.[9] Also, dynamical mass generation which is the correct phase for these models in two dimensions occurs naturally at the lowest order in $1/N$. For non-Abelian gauge theories, it has been argued that the $1/N$ expansion is consistent with confinement even if it has not been proved in that scheme.[10] The importance of the expansion for the strong interactions may be not that $\frac{1}{3}$ is such a small parameter but rather that the $1/N$ formulation starts out in the correct phase for the four-dimensional theory.

On the other hand, the perturbation singularities seem to give more information about the short-distance behavior of hard processes such as deep-inelastic lepton scattering and the Drell-Yan mechanism. It has not been proved that an operator-product expansion exists in the $1/N$ scheme. There may be a sort of complementarity in which perturbation theory is best for studying factorization and scaling violations in hard processes, and $1/N$ is best for understanding confinement, spectra, and on-mass-shell interactions.

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