Fractal Boundary for Chaos in a Two-State Mechanical Oscillator

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Measurements of chaotic vibrations of a forced nonlinear oscillator with a two-well potential suggest that the frequency-amplitude boundary between periodic and chaotic motions may be fractal. The fractal dimension of this curve is 1.26. The experiment consisted of an elastic beam and magnets which were used to create two stable equilibrium positions. Analysis has shown that this system possesses strange-attractor solutions for a single-mode model. However, a fractal boundary suggests that the role of higher modes may be important.

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Nonlinear dynamics in a classical two-well potential have been studied as a model for plasma oscillation by Mahaffey[1] and for vibrations of a buckled elastic beam by Holmes.[2] In previous studies[3,4] we reported strange-attractor vibrations of a buckled nonlinear elastic beam. This system has two stable and one unstable equilibrium points as shown in the zero-damping phase portrait in Fig. 1. Forced oscillations of this system have been studied analytically, numerically,[2] and by experiment.[3,4] This model of a strange attractor is one of the few that has received both extensive experimental as well as theoretical analysis. The mathematical model using a one degree-of-freedom approximation has the form

$$\ddot{A} + \gamma\dot{A} - \tfrac{1}{2}(1-A^2)A = f\cos\omega t, \tag{1}$$

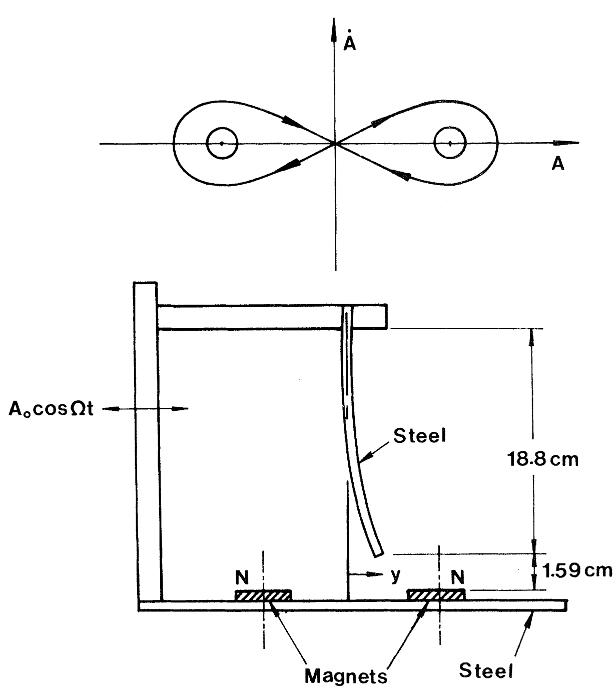

FIG. 1. Top: Phase plane portrait of separatrix for the undamped, unforced system. Bottom: Sketch of experimental apparatus showing steel beam and two magnets.

where A is the modal amplitude of the first vibration mode and (1) is based on a Galerkin projection of the partial differential equations for the flexible beam. Extensive study of this system has shown that it possesses chaotic solutions for given parameters (γ, f, ω) and initial conditions. Experimental observations of this system do not exhibit a periodic-doubling route to chaos, however. Holmes[2] has derived a necessary condition for chaos which takes the form of a smooth function of (γ, f, ω):

$$f \geqslant (\gamma\sqrt{2}/3\pi\omega)\cosh(\pi\omega/\sqrt{2}). \tag{2}$$

The author has also derived a more heuristic condition based on the velocity necessary to jump out of the energy well[4]:

$$f \geqslant \frac{\alpha}{2\omega}\left[\left(1-\omega^2-\frac{3}{8}\frac{\alpha^2}{\omega^2}\right)^2+\gamma^2\omega^2\right]^{1/2}, \tag{3}$$

where α is a parameter close to 1. Experimental measurements[3] of the critical forcing amplitude between chaotic motion and periodic motions, however, revealed a nonsmooth boundary with driving frequency as a parameter. This variability was much greater than experimental errors in the system. At least five sets of data for chaotic boundaries were taken for different magnet-beam configurations and all showed a nonsmooth behavior. A Poincaré map of this system can be obtained by measuring $(A, \dot{A})$ at a certain phase of the driving force in (1). These experimental maps showed a stable pattern over many hours (Fig. 2).

Recent papers on the fractal nature of chaotic motions have encouraged the author to speculate as to whether the boundary is fractal. Grebogi, Ott, and Yorke[5] have observed fractal boundaries of ini-

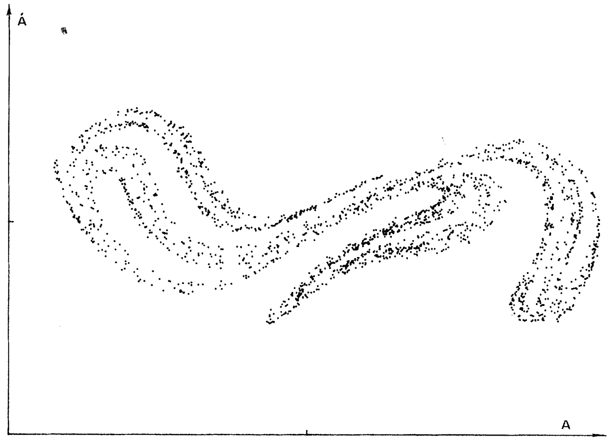

FIG. 2. Experimental Poincaré map for chaotic vibrations of a buckled beam.

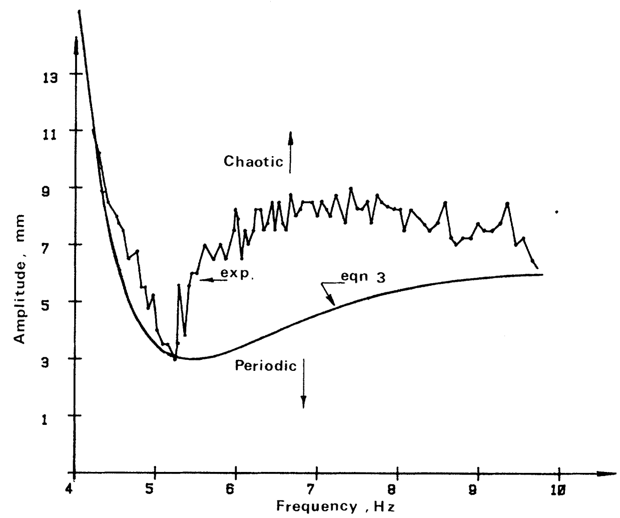

FIG. 3. Comparison of experimental and theoretical boundaries between periodic and chaotic vibrations.

tial conditions for a two-dimensional map which leads to chaos in numerical simulation. Malraison *et al.*[6] have measured the fractal dimension of the chaotic time history of temperature in Rayleigh-Bénard convection.

In this paper I propose that the boundary between chaotic and periodic oscillations in a parameter space is fractal. This behavior is believed to be related to the higher vibration modes in the beam which are neglected in the one-mode model expressed in (1).

The experiment has been described in Refs. 2 and 3. The clamped end of the beam is driven by an electromagnetic vibrator where $f = -\omega^2 A_0$. With the beam vibrating initially with periodic motion in one of the potential wells, the amplitude was increased until the beam jumped out of the well. To determine whether the motion was periodic or chaotic, an experimental Poincaré map (i.e., a set of points in the $A,\dot{A}$ plane) was used which was synchronized with the forcing amplitude. Chaos was determined when a finite set of points (subharmonic motion) became unstable and a Cantor-set–like pattern appeared on the screen similar to that in Fig. 2. All measurements were made for increasing frequency. However, on the basis of a few measurements, I believe that the criterion is hysteretic with regard to increasing or decreasing amplitude.

In an earlier experiment,[3] approximately 35 frequencies were explored in a domain between 5–10 Hz. Three other sets of data showed similar behavior. At that time we did not suspect that the boundary might be fractal, on the basis of the analysis leading to (2) and (3). In this recent experiment we measured the boundary at approximately 70 frequencies between 4–9 Hz. Typical of all measurements of the criterion for chaos are the data shown in Fig. 3. This is in contrast to the smooth boundaries predicted by Moon[4] and Holmes.[2,3]

To measure the fractal dimension of the experimental boundary for chaos, in the (A_0, ω) plane the curve between two points P_1 and P_2 is approximated by a connected set of N straight lines of length l. The length of the approximate curve is then $L = Nl$. As $l \to 0$, $N \to \infty$. For a smooth or nonfractal curve $N \to \lambda l^{-1}$ and we call λ the length. However, for a fractal curve $N \to \lambda l^{-D}$ so that

$$L = \lambda l^{1-D}. \tag{4}$$

When D is not an integer the curve is called fractal. This definition is identical to that of the capacity of a set.[7] In this experiment a drafter's caliper was used to measure the length of the curves in Fig. 3 for ten different caliper lengths. This simple technique for determining the fractal dimension of a curve is described by Mandelbrot[8] for the problem of measuring the length of coast lines. These data are shown plotted in Fig. 4 for two sets of data—one set reported in Ref. 3, and the other with twice the number of measurements conducted for this study. Note that at the time we reported data in Ref. 3, we did not suspect a fractal curve. The total length shows a behavior characteristic of fractal sets, namely, an increase in length with a increase in l. The two slopes of the curves in Fig. 4 as determined by a least-squares fit lead to fractal dimensions of $D = 1.24$ and 1.28.

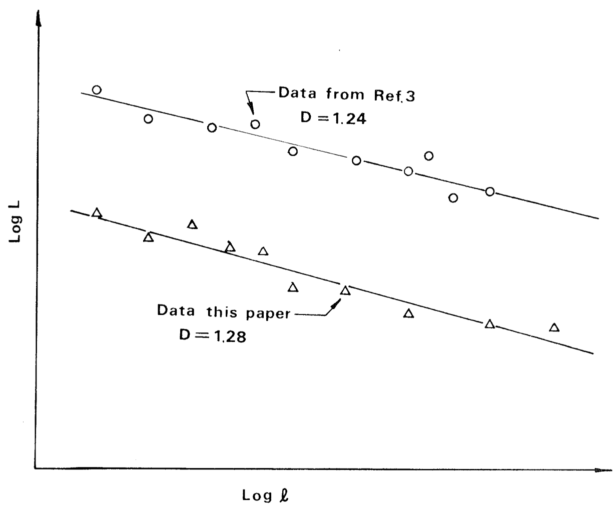

FIG. 4. Total length of the experimental boundary curve, L, as a function of caliper length, l.

Ott and co-workers at the University of Maryland have argued that the minimum dimension for chaos for an invertible map is two, and for a fractal basin boundary (for initial conditions) one must have a three-dimensional map. For the forced Duffing equation (1), the Poincaré map of the system of differential equations is two dimensional (Fig. 2) and this leads to the question of whether fractal behavior in the parameter space is to be expected. In fact, numerical simulation of (1) seems to support Ott's conjecture that a smooth boundary should result. In the experimental system, however, we may resolve the problem by recognizing that the physical system is really of infinite dimension, although a one-mode projection seems to reproduce all the qualitative behavior—including chaos. This suggests that the higher modes of vibration are required to explain the fractal nature of the chaos criterion. One could use the first N $(N \geqslant 2)$ free-vibration modes of the elastic beam to describe the motion. The equation of motion for the transverse deflection of the beam $v(x,t)$ is given by[3,9]

$$\ddot{v} + a^2 v'''' + b^2 v'' - [cv(L) + dv^3(L)]\delta(x-L) = -\ddot{V}(t), \qquad (5)$$

where $\ddot{V}(t)$ is the harmonic motion of the clamped base and the fourth term on the left side of (5) represents the magnetic force components at the tip of the beam.[3] Using Galerkin's method one can derive two coupled ordinary differential equations for the modal amplitudes. The Poincaré map of this system would be four dimensional or higher which would satisfy the criterion of Ott for fractal boundaries.

The erratic nature of the boundary between chaotic and periodic motion is much larger than the experimental errors in the measurements. Given the known fractal nature of the strange attractor itself and basin boundaries there is convincing evidence that the chaos boundary measured in this experiment is fractal and is related to the influence of higher modes. Possible implications of these results are that parameter space boundaries between chaotic or turbulent and periodic motions in continuous systems may be fractal. Thus a clear-cut criterion may not be possible as is often found experimentally in fluid mechanics. Variability in the critical parameter for chaos or turbulence may be inherent in many continuous systems and may not be solely related to experimental errors.

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[1]R. A. Mahaffey, Phys. Fluids **19**, 1387–1391 (1976).

[2]P. J. Holmes, Philos. Trans. Roy. Soc., London **292**, 419–448 (1979).

[3]F. C. Moon and P. J. Holmes, J. Sound Vibr. **65**, 276–296 (1979).

[4]F. C. Moon, J. Appl. Mech. **47**, 638–644 (1980).

[5]C. Grebogi, E. Ott, and J. A. Yorke, University of Maryland Plasma Report No. PL83-035, 1982 (unpublished).

[6]B. Malraison, P. Atten, P. Berge, and M. DuBois, to be published.

[7]J. D. Farmer, E. Ott, and J. A. Yorke, Physica (Utrecht) **7D**, 153 (1983).

[8]B. Mandelbrot, *Fractals* (Freeman, San Francisco, 1977).

[9]P. J. Holmes and J. E. Marsden, Arch. Ration. Mech. Anal. **76**, 135–166 (1981).