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Analysis of the Experiment to Determine the Spectrum of the Angular Momentum of a Charged-Boson, Magnetic-Flux-Tube Composite and the Aharonov-Bohm Effect

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By means of the Feynman path-integral method the experiment proposed by Silverman is analyzed under the classical approximation. The results give a new verification of the Aharonov-Bohm effect. It is pointed out that the multiple-valued wave functions are reasonable for describing the experiment. Furthermore, it is verified that the eigenvalues of the angular momentum of a charged, spinless particle-solenoid composite are flux dependent.

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Wilczek[1] recently has explored subtle quantum properties of a system containing electric charge and a solenoid. According to him, the rotation generator of a charged, spinless particle orbiting a current-carrying impenetrable solenoid can be given arbitrary eigenvalues. Jackiw and Redlich[2] point out that there is a difference between the kinetic angular momentum and the canonical angular momentum whenever velocity-dependent forces occur. The focal point of their arguments concerns whether the kinetic or canonical angular momentum connects with the spin statistics.

Among various arguments, there is an interesting paper by Silverman[3] proposing an experiment to test for the rotational properties of the charged, spinless-particle–magnetic-flux-tube composite.

In the present paper we will first analyze the experiment using the Feynman path-integral method. We will then use multiple-valued wave functions to describe the experiment under rotations.

Following Silverman, an incident wave packet $\psi(\vec{x}_0,t_0)$ at time t_0 splits into two coherent wave packets separately passing around two current-carrying impenetrable solenoids. At time t the two wave packets meet at a detector.

The wave function at time t according to Feynman is

$$\psi(\vec{x},t)=\int d^3\vec{x}_0\,K(\vec{x},t;\vec{x}_0,t_0)\psi(\vec{x}_0,t_0), \tag{1}$$

where the propagator $K(\vec{x},t;\vec{x}_0,t_0)$ from $\vec{x}_0,t_0$ to $\vec{x},t$ is given by the path integrals

$$K(\vec{x},t;\vec{x}_0t_0)=\sum\exp\left\{\frac{i}{\hbar}\int_{t_0}^{t}L(\vec{x},\dot{\vec{x}},t')\,dt'\right\}. \tag{2}$$

The infinite sum is over all of possible paths in space-time from $\vec{x}_0$ to $\vec{x}$. The Lagrangian L in the discussed case is

$$L=\tfrac{1}{2}\mu\dot{\vec{x}}^2+(q/c)\vec{A}(\vec{x},t')\cdot\dot{\vec{x}}. \tag{3}$$

As a result of the arrangement of the experiment, the classical approximation can be used.[4] For the two beams of the charged particles the wave function at time t is

$$\psi(\vec{x},t)=\psi_R(\vec{x},t)+\psi_L(\vec{x},t), \tag{4}$$

where the ψ_R and ψ_L represent the wave packets correspondingly orbiting the right-hand and the left-hand solenoid. For

$$\psi_R(\vec{x},t) \sim \exp\left\{\frac{i\mu}{2\hbar}\int \dot{\vec{x}}\cdot d\vec{x} + \frac{iq}{c\hbar}\int \vec{A}^{(R)}\cdot d\vec{x}\right\}, \tag{5}$$

the integral is evaluated along the right-hand path, while for

$$\psi_L(\vec{x},t) \sim \exp\left\{\frac{i\mu}{2\hbar}\int \dot{\vec{x}}\cdot d\vec{x} + \frac{iq}{c\hbar}\int \vec{A}^{(L)}\cdot d\vec{x}\right\}, \tag{6}$$

the integral is evaluated along the left-hand path. In the above

$$\vec{A}^{(R)} = \frac{\alpha\phi_0}{2\pi r}\theta - \frac{B}{2}r\theta, \quad \vec{A}^{(L)} = -\frac{\alpha\phi_0}{2\pi r}\theta - \frac{B}{2}r\theta, \tag{7}$$

where $\alpha\phi_0$ is the flux through the solenoid $\phi_0 = ch/q$; B is the background magnetic field and θ is the unit vector of θ direction.

Working out the integrals in Eqs. (5) and (6) along the classical orbits in n revolutions we have

$$\psi = \psi_R + \psi_L \sim [\exp(i2n\pi\alpha) + \exp(-i2n\pi\alpha)]. \tag{8}$$

The intensity is

$$I \sim \cos^2 2n\pi\alpha. \tag{9}$$

This result gives the familiar Aharonov-Bohm interference: The intensity depends on the flux in the solenoid. If one adjusts the flux so that $\alpha = \frac{1}{2}$ it leads to a constructive interference.[5]

Thus our interference is an Aharonov-Bohm-like result. According to Silverman the experiment should confirm that the kinetic angular-momentum operator should serve as the generator of rotations. On the other hand, Jackiw and Redlich[2] have already demonstrated that the generator of rotations is the canonical angular momentum prescribed by Noether's theorem as a conserved quantity. This contradiction can be appropriately solved by using the canonical angular momentum as the generator of rotations as Jackiw and Redlich do. However, we will use multiple-valued wave functions for the charged particle and impenetrable solenoid system.[6]

In Silverman's experiment the particles move along a circular orbit under the influence of the background field. In the classical limit the problem reduces to a plane rotator.

In a nonsingular gauge, the Hamiltonian of the plane rotator orbiting the solenoid is given by

$$H = (\hbar^2/2I)[-i(\partial/\partial\theta) - \alpha]^2, \tag{10}$$

where $I = \mu R^2$ is the moment of inertia and R the radius of the orbit.

For single-valued wave functions,

$$\psi = \sum C_m \psi_m, \quad \psi_m = e^{im\theta}. \tag{11}$$

The eigenvalues of energy are given by

$$E_m = (\hbar^2/2I)(m-\alpha)^2. \tag{12}$$

This result is considered as the bound-state Aharonov-Bohm effect in the literature.[7]

For the multiple-valued wave functions, which should satisfy the Bloch condition $\psi|_{\theta+2\pi} = e^{i\lambda}\psi|_\theta$ (where in our case $\lambda = 2\pi\alpha$),

$$\psi_m = e^{im\theta}e^{i\alpha\theta}, \tag{13}$$

where m is integer.

The eigenvalues of the energy are

$$E_m = (\hbar^2/2I)m^2, \tag{14}$$

while the angular momentum and kinetic angular momentum are given by

$$l_z = m + \alpha, \quad l_z^{\text{kin}} = m. \tag{15}$$

The eigenvalues of the energy and kinetic angular momentum are the same as for a flux-free rotator.

Under a singular gauge transformation[2]

$$\vec{A}' = \frac{\alpha\phi_0}{2\pi r}\hat{\theta} - \nabla\Lambda, \quad \Lambda = \frac{\alpha\phi_0}{2\pi}\theta, \tag{16}$$

which corresponds to a unitary transformation in quantum mechanics

$$U = \exp\left(-\frac{q\Lambda}{c\hbar}\right) = \exp(-i\alpha\theta), \tag{17}$$

the canonical angular momentum and wave functions become

$$L_z' = UL_zU^{-1} = \hbar[-i(\partial/\partial\theta) + \alpha], \tag{18}$$

$$\psi_m' = U\psi_m = e^{im\theta}. \tag{19}$$

Any eigenvalue of observables is gauge invariant. In the singular gauge the wave functions are just formal single-valued functions, but they are not invariant under a 2π "rotation"

$$R'(2\pi)\psi_m' = \exp[-i(L_z'/\hbar)2\pi]e^{im\theta} = e^{im\theta}e^{-i2\pi\alpha}. \tag{20}$$

The wave functions in Eq. (13) were used to argue against the Aharonov-Bohm effect.[8] However, the multiple-valuedness of the wave functions, which incur a phase change under a 2π rotation, naturally causes the Aharonov-Bohm interference. This has been noticed by Henneberger.[9]

In the following, Silverman's experiment will be reanalyzed using the multiple-valued wave functions for the reason that the mean angular velocity of a wave packet constructed by the superposition of multiple-valued eigenfunctions (13) is independent of α in agreement with the correspondence principle under the classical limit. As a matter of fact, the mean angular velocity of the wave packet equals $qB/\mu c[=(2\hbar/I)\alpha_B]$ where $\alpha_B\phi_0$ is the flux of the background magnetic field embraced by the orbit.[10]

For the right-hand path the Hamiltonian and the wave functions are given by

$$H^R=(\hbar^2/2I)[-i(\partial/\partial\theta)-\alpha]^2, \tag{21}$$

$$\psi_m^R=e^{im\theta}e^{i\alpha\theta}, \tag{22}$$

while for the left-hand path

$$H^L=(\hbar^2/2I)[-i(\partial/\partial\theta)+\alpha]^2, \tag{23}$$

$$\psi_m^L=e^{im\theta}e^{-i\alpha\theta}. \tag{24}$$

For the sake of simplicity the background field is not accounted for here.[3] The eigenvalues of the energy are the same for the right- and left-hand paths, since the radius of the orbit is the same for both sides. These eigenvalues are

$$E_m^R=E_m^L=(\hbar^2/2I)m^2. \tag{25}$$

A wave packet can be constructed by the superposition of eigenfunctions:

$$\psi^R(\theta,t)=\sum_{m=-\infty}^{\infty}C_m^R e^{im\theta}e^{i\alpha\theta}\exp-i(E_m^R/\hbar)t. \tag{26}$$

After the wave packet moves around the solenoid n revolutions, the wave function becomes[11]

$$\psi^R(\theta+n2\pi,t+n\tau)=\exp i(L_z/\hbar)^{n2\pi}\psi^R(\theta,t+n\tau)=\psi^R(\theta,t+n\tau)e^{i2\pi n\alpha} \tag{27}$$

where τ is the period of one revolution. With the expression in (26) we find

$$\psi^R(\theta,t+n\tau)=\sum_{m=-\infty}^{\infty}C_m^R e^{im\theta}e^{i\alpha\theta}\exp-i(E_m^R/\hbar)(n\tau+t). \tag{28}$$

In a similar way the wave function on the left hand is

$$\psi^L(\theta+n2\pi,t+n\tau)=\psi^L(\theta,t+n\tau)e^{-i2\pi n\alpha}. \tag{29}$$

Then the forward wave packet is given by

$$\psi=\psi^R+\psi^L\sim e^{i2\pi n\alpha}+e^{-i2\pi n\alpha}, \tag{30}$$

and

$$I\sim\cos^2 2n\alpha\pi. \tag{31}$$

This agrees with the result (9) obtained using the Feynman-path-integral method. This Letter reports the first use of an explicit multiple-valued wave function to analyze an Aharonov-Bohm interference experiment.

In conclusion, our results give the familiar Aharonov-Bohm interference. We also verify, under the classical limit, that the orbital angular-momentum spectrum of a charged, spinless particle in a multiply connected space is quantized in units

$$l_z=\text{integer}+\alpha. \tag{15}$$

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[5]It is not necessary for the flux through each solenoid to be equal. For the general case of flux $\phi_R=\alpha_R\phi_0$ through right-hand solenoid and $\phi_L=\alpha_L\phi_0$ through left-hand solenoid, the forward wave packet is given by $\psi=\psi_R+\psi_L\sim\exp(i2\pi n\alpha_R)+\exp(-i2\pi n\alpha_L)$. If we choose $n=1$, $\alpha_L=0$, then $\alpha_R=\frac{1}{2}$ will give complete destructive interference. See also the recent paper by M. P. Silverman, Phys. Rev. D **29**, 2404 (1984).

[6]For a multiply connected space, there is no single-valuedness requirement on the wave functions. See Ref. 2.

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[10]As an example, let us consider a Gaussian wave packet $\psi(\theta)=N\exp-[\theta^2/2(\Delta\theta)^2]\,e^{i\alpha\theta}\exp(i\alpha_B\theta)$, where $\exp[-\theta^2/2(\Delta\theta)^2]$, with a 2π periodic extension, is defined in the region $-\pi\leq\theta\leq\pi$. The mean angular velocity of the wave packet is evalued by $\langle\psi|(\hbar/I)[-i(\partial/\partial\theta)-\alpha+\alpha_B]|\psi\rangle=(2\hbar/I)\alpha_B$, in agreement with the classical angular velocity.

[11]In the singular gauge [Eq. (16)] the rotation operator is $R'(n2\pi)=\exp[-i(L_z'/\hbar)n2\pi]$, so that under a $n2\pi$ rotation the wave function incurs the same phase change $(e^{i2\pi n\alpha})$ which leads to the Aharonov-Bohm interference.