Critical Behavior of a Dilute Interacting Bose Fluid

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The crossover from critical behavior to ideal Bose-gas behavior in a dilute, weakly interacting Bose fluid at low temperatures is shown to be described by a scaling function depending on the temperature, density, and dimensionality. The results are applied to the experimental measurements of Crooker *et al.* on the superfluid density of ^{4}He in Vycor glass. The satisfactory agreement obtained lends support to the model of ^{4}He in Vycor as a dilute, weakly interacting, three-dimensional Bose fluid.

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In this Letter we study Bose-Einstein condensation in a dilute, weakly interacting Bose fluid and show that the crossover from critical behavior to ideal Bose-gas behavior can be described by a scaling function. Possible realizations of such a system are provided by superfluid ^{4}He films adsorbed on porous Vycor glass[1] and by spin-polarized hydrogen.[2] In this Letter we address ^{4}He in Vycor, which is a highly connected, spongelike glass with a pore size of 50–80 Å in the experiments of Crooker *et al.*[1] These authors have studied the superfluid density in Vycor over a wide range of overall density ρ, the resulting superfluid transition temperatures varying from 2 K down to 4.4 mK at the lowest density studied. For all critical temperatures, $T_c(\rho)$, the experiments are consistent with the superfluid density varying as $\rho_s \sim (T_c - T)^\zeta$ as $T \rightarrow T_c$ with $\zeta \simeq \frac{2}{3}$. Thus for all ρ the superfluid transition in Vycor appears similar to that occurring in bulk ^{4}He, the exponent ζ being sensibly constant. This suggests that the disorder introduced by the irregular Vycor structure is irrelevant to the critical behavior. This is consistent with the general theory of the effects of disorder[3] and the fact that the specific heat exponent of bulk ^{4}He is negative ($\alpha \simeq -0.02$). The thermal wavelength of the ^{4}He atoms is comparable with the pore size at the lower temperatures which should reduce the effects of disorder. Vycor is a highly connected structure[4] and although the ^{4}He atoms are confined to the surface by the strong van der Waals potential, they would be expected to behave as a three-dimensional fluid.[5] At the lower densities the superfluid interparticle spacing is from 5 to 12 times the atomic hard-core diameter and thus the fluid is very dilute.

The observed size of the critical region described by $\zeta \simeq \frac{2}{3}$ shrinks as ρ decreases and a crossover to ideal Bose-gas or mean-field-like behavior, $\rho_s \sim T_c - T$, is seen as $T_c(\rho) \rightarrow 0$. The scaling theory of critical phenomena indicates that this behavior should be described by a scaling function, the existence of which should be independent of any detailed model of ^{4}He in Vycor. The theory discussed below shows that the form of the scaling function for the superfluid density at low temperatures and densities in d dimensions is

$$\rho_s = \rho_s(0)(T/T_c)^{d/2} t Y[(a/\Lambda_T)^{d-2}/t^\phi]. \qquad (1)$$

Here $\rho_s(0)$ is the superfluid density at $T=0$, $t=(T_c/T)^{d/2}-1$, a is the scattering length characterizing the interaction of He atoms, $\Lambda_T = (h^2/2\pi m^* k_B T)^{1/2}$ is the thermal wavelength of ^{4}He atoms of effective mass m^*, and ϕ is a universal crossover exponent. Ideal Bose-gas behavior follows when $T \rightarrow 0$ if $Y(0)=1$ and a crossover to bulk behavior, $\rho_s \sim t^\zeta$, follows as $t \rightarrow 0$ at $T_c > 0$ if $Y(y) \sim y^{(1-\zeta)/\phi}$ as $y \rightarrow \infty$. The critical region thus shrinks like $t \sim T^{(d-2)/2\phi}$ as $T_c \rightarrow 0$. The aim of this Letter is to test (1) against the data, to determine the exponent ϕ, and to report a calculation of the scaling function, $Y(y)$, to order $\epsilon = 4-d$. In fact we find

$$\phi = (4-d)/(d-2) \qquad (2)$$

for all $2 < d \leq 4$. For $d=3$ this yields $\phi = 1$ and Fig. 1 then demonstrates that the data of Crooker *et al.*[1] satisfy (1) rather well. The calculated scaling function (solid curve) gives a good qualitative fit.

We approach the problem by modeling ^{4}He in Vycor as a mobile, low-density, spinless Bose gas in a d-dimensionally connected medium whose properties enter only through an effective mass, m^*, and an effective two-body potential with Fourier transform v_k. The Hamiltonian is

$$\mathcal{H}_{\rm B} = \sum_k \epsilon_k a_k^\dagger a_k + \frac{1}{2V} \sum_{kK'k''} v_k a_{k'}^\dagger a_{k''}^\dagger a_{k'+k} a_{k''-k}, \tag{3}$$

where $\epsilon_k = \hbar^2 k^2/2m^*$ and $a_k^\dagger$, a_k are Bose creation and annihilation operators. At low densities we can replace v_k by an effective s-wave scattering potential, v_0, which for $d=3$ is $v_0 \sim (\hbar^2/2m^*) a^{d-2}$ where a is the scattering length.[6] Both m^* and a should be only weakly temperature-dependent.

In order to study the superfluid transition and utilize current knowledge of critical phenomena, we map (3) onto a classical spin model. To this end we first note that the critical properties of an ideal Bose gas coincide with those of the spherical model of a ferromagnet.[7] Secondly, for $T_c \neq 0$, critical behavior does not depend on the noncommutative aspects of quantum mechanics.[8] We thus compare (3) with

$$\mathcal{H}_S = -\tfrac{1}{2} \sum_k J_k |\vec{S}_k|^2 + \frac{1}{V} \sum_{kk'k''} u_k \vec{S}_{-k'} \cdot \vec{S}_{k+k'} \vec{S}_{-k''} \cdot \vec{S}_{k''-k}, \tag{4}$$

where the exchange coupling, of finite range R_0, is $J_k \simeq J_0(1 - R_0^2 k^2 + \ldots)$ while the $\vec{S}_k$ are classical n-component vectors with $-\infty < \vec{S}_k^\mu < \infty$. Corresponding to the implied constant-density constraint for (3) we impose the spherical constant

$$\frac{1}{nV} \sum_k |\vec{S}_k|^2 = \rho = \frac{1}{V} \sum_k \langle a_k^\dagger a_k \rangle. \tag{5}$$

For $u_k = 0$, (4) and (5) specify the spherical model. In order to match the ideal Bose gas properly in the critical region, an appropriate wave vector cutoff, k_Λ, is required in (4). One expects

$$k_\Lambda = C_d/\Lambda_T \equiv C_d(2\pi m^* k_{\rm B} T/h^2)^{1/2}, \tag{6}$$

where Λ_T is the thermal wavelength. This is confirmed by matching the critical free energies of the two models which further yields

$$J_0 R_0^2 = \hbar^2/2m^*,$$
$$C_d^{d-2} = \tfrac{1}{2}(4\pi)^{(d-2)/2}(d-2)\Gamma(d/2)\zeta(d/2). \tag{7}$$

Here $\Gamma(z)$ and $\zeta(z)$ are the gamma and zeta functions and we have set $n=2$ (see below). The interaction parts of (3) and (4) can likewise be matched by comparing the corresponding perturbation expansions. These have identical structures if, but only if, $u_k = v_k/8$ and $n = 2$.[9]

The constraint (5) can be handled by adding a term $-\frac{1}{2} z \sum_k |\vec{S}_k|^2$ to (4) where z is analogous to the chemical potential. The resulting reduced-spin Hamiltonian with momenta rescaled to unit cutoff is then

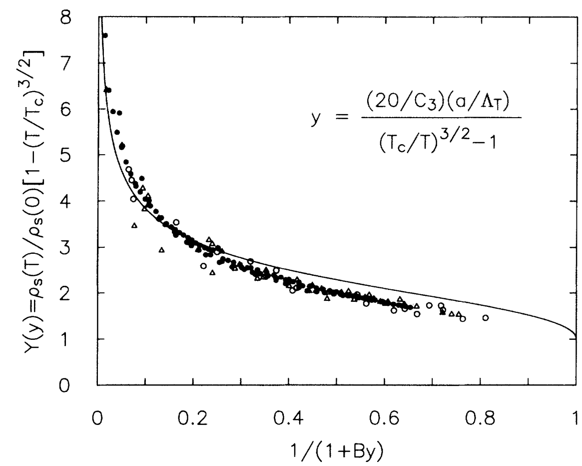

FIG. 1. A scaling plot of the experimental data (Ref. 1), including eleven values of $T_c(\rho)$ from 6 to 12 mK (open circles), 15 to 35 mk (triangles), and 37 To 77 mK (filled circles): To plot the data we choose, purely for convenience, to set $(a^2 m^*/m_{\rm He})^{1/2} = 5$ Å and $B = 0.025$. The solid line is the theoretical result, the single adjustable parameter being fixed by choosing $(a^2 m^*/m_{\rm He})^{1/2}_{\rm theo.} = 200$ Å to match the $y^{1-\nu}$ singularity at the origin, $y \to \infty$ (Ref. 9).

$$\bar{\mathcal{H}}_\sigma = \tfrac{1}{2} \int_q (r + q^2) \vec{\sigma}_q \cdot \vec{\sigma}_{-q} + u \int_q \int_{q'} \int_{q''} \vec{\sigma}_{-q'} \cdot \vec{\sigma}_{q'+q} \sigma_{-q''} \cdot \vec{\sigma}_{q''-q}, \tag{8}$$

where $\int_q \equiv \int d^d q/(2\pi)^d$, while

$$r = -(z + J_0)/J_0 R_0^2 k_\Lambda^2, \quad u = u_0 k_\Lambda^{d-r} k_{\rm B} T / J_0^2 R_0^4. \tag{9}$$

Note that u varies as $(a^2T)^{(d-2)/2} \sim (a^2T_c)^{(d-2)/2}$ in the critical region. Ideal-gas behavior ($v_0=0$) corresponds to the standard Gaussian fixed point, G, about which u is a relevant renormalization-group perturbation scaling as t^{ϕ_G} with $\phi_G=\epsilon/2$ (for all d). This, however, neglects the constraint (5) which now becomes

$$\frac{\partial F}{\partial r}=\frac{n(T_c^0/T)^{d/2}}{2^d\pi^{d/2}(d-2)\Gamma(d/2)}, \qquad (10)$$

where F is the free-energy density for (8) while T_c^0 is defined by $\rho\Lambda^d(T_c^0)=\zeta(d/2)$. This constraint leads to exponent renormalization[10] in the usual way so that ϕ_G is replaced by $\phi=\phi_G/(1-\alpha_G)$ where $\alpha_G=\epsilon/2$, which is the result (2).

The Hamiltonian (8) has been studied extensively and the free energy has been calculated by Rudnick and Nelson[11] and Nicoll and Chang[12] to order ϵ. The onset of the critical region is where the argument of the scaling function (1) is of order unity; more quantitatively from Ref. 11 we have

$$\tfrac{1}{2}\epsilon t \simeq (u/u^*)^{1/\phi}=(T/T_0)^{(d-2)/2\phi}, \qquad (11)$$

where $u^*=2\pi^2\epsilon/(n+8)$ is the fixed point value of u. The parameter T_0 is $(C_3u^*/\pi a)^2(h^2/8\pi m^* k_B)$ in three dimensions. Taking $a=3$ Å, $m^*=m_{He}$, and $C_3=4.2$ we find $T_0\simeq 0.3$ K. Replacing T by T_c on the right in (11) shows that the critical region varies with density as $t\sim\rho^{(d-2)2/d(4-d)}$ or $t\sim\rho^{1/3}$ in $d=3$.

The experiments of Crooker *et al.*[1] determine the superfluid density whose magnetic analog[13] is the helicity modulus Υ which has been calculated to order ϵ by Jasnow and Rudnick[14]:

$$\Upsilon/kT=-(\bar r/4u)Q^{6/(n+8)}[1+(u^*/4\pi^2)(1-\bar Q^{-1})]k_\Lambda^{d-2}, \qquad (12)$$

where $\bar r = r+(n+2)u/4\pi^2+O(\epsilon u,u^2)$, $\tilde u=u/u^*$, $\bar Q=Q(1-\tilde u)^{-1}$, and Q satisfies

$$Q=1-\tilde u+\tilde u(-2\bar r)^{-\epsilon/2}Q^{g_n\epsilon}, \qquad (13)$$

with $g_n=(n+2)/2(n+8)$. From the free energy of Ref. 12 the constraint (10) becomes[9]

$$\frac{\bar r}{2(n-4)u}\left\{Q^{(4-n)/(n+8)}\left[\frac{1-\frac{1}{4}\epsilon(1-\bar Q^{-1})}{1-g_n\epsilon(1-\bar Q^{-1})}\right]\frac{n}{4}\right\}=\frac{nt}{(4\pi)^{d/2}\Gamma(d/2)(d-2)}. \qquad (14)$$

Equations (12), (13), and (14) allow Υ to be expressed in terms of temperature and density by eliminating $\bar r$ in favor of t.

The scaling form (1) follows by writing (13) and (14) in scaling form.[9] Thus (13) has the solution $\bar Q=Z(\dot u/\dot r^{\epsilon/2})\equiv Z(x)$ where $\dot u=\tilde u(1-\tilde u)^{-1}$ and $\dot r=-2r(1-\tilde u)^{-2g_n}$ are nonlinear scaling fields. The constraint (14) can be written

$$\frac{n}{(4\pi)^{d/2}\Gamma(d/2)(d-2)}\frac{\dot t}{\dot u^{1/\phi}}=\frac{x^{-2/\epsilon}}{4(4-n)u^*}\left[Z(x)^{(4-n)/(n+8)}\left(\frac{1-\frac{1}{4}\epsilon[1-Z(x)^{-1}]}{1-g_n\epsilon[1-Z(x)^{-1}]}\right)-\frac{n}{4}\right] \qquad (15)$$

where $\dot t=t(1-\tilde u)^{2g_n}$ and we have omitted a small term on the right. We write the solution of (15) in the form $x=T(y)$ where $y=\dot u/\dot t^{\phi}$. The helicity modulus (12) can be written

$$\Upsilon/k_BT=\dot t Y(y)\{(2nk_\Lambda^{d-2})/[(d-2)(4\pi)^{d/2}\Gamma(d/2)]\}, \qquad (16)$$

where

$$Y(y)=y^{1/\phi}\{U[T(y)]/T^{2/\epsilon}(y)\}[(d-2)\Gamma(d/2)(4\pi)^{d/2}/2n], \qquad (17)$$

and from (12) U satisfies

$$U(x)=\frac{1}{8u^*}Z(x)^{6/(n+8)}\left\{1+\frac{u^*}{4\pi^2}[1-Z^{-1}(x)]\right\}. \qquad (18)$$

Finally, inverting the rescaling transformation leading to (8) gives the superfluid density

$$\rho_s=\left(\frac{m^*}{\hbar}\right)^2\Upsilon=\frac{n}{2}\rho m^*\left(\frac{T}{T_c}\right)^{d/2}\dot t Y(\dot u/\dot t^{\phi}), \qquad (19)$$

which at low temperatures and close to T_c is of the form (1). In the limit $y\to 0$ ($\dot u \ll \dot t^{\phi}$), $Y(0)=1$, and ρ_s reduces to the spherical model result. In the opposite limit $y\to\infty$ ($\dot t^{\phi}\ll \dot u$) we find $\rho_s\sim \dot t^{v/(1-\alpha)}$ where $v=(1-\frac{1}{2}\epsilon)/(1-g_n\epsilon)$ is the helicity-modulus critical exponent and $\alpha=\epsilon(4-n)/[2(n+8)(1-g_n\epsilon)]$ is the specific heat exponent, both to order ϵ.

In conclusion, the crossover from critical to ideal

Bose gas of the superfluid density has been shown to be described by a scaling function. This scaling function is in satisfactory agreement with the experimental measurements on ^{4}He in Vycor, supporting the description of this system as a dilute, weakly interacting, three-dimensional Bose fluid.

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