Predicting the Proton Mass from $\pi\pi$ Scattering Data

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We relate experimental information on $\pi\pi$ scattering to tree-level effective chiral Lagrangians. The result is of a form similar to that used in Skyrme-type models of the proton, where the nucleons are described as topologically stable solitons of the chiral fields. In such models, one can express the proton mass in terms of measured scattering data, with the result $M_p = 880 \pm 300$ MeV. We interpret this as a consistency test for the Skyrme models.

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Quantum chromodynamics[1] has an approximate chiral SU(2) symmetry[2]

$$\begin{pmatrix} u \\ d \end{pmatrix} \to \exp(i\vec{\alpha}\cdot\vec{\tau}\gamma_5)\begin{pmatrix} u \\ d \end{pmatrix}, \tag{1}$$

which is broken only by the small (current) masses of the up and down quarks. This symmetry appears to be dynamically broken, implying that if the quark masses were zero, the pions would be massless Goldstone bosons. At very low energies the world would then be describable in terms of the pion degree of freedom with interactions dictated by the chiral invariance. Even in the realistic situation with nonzero quark and pion masses, PCAC (partial conservation of the axial current) techniques can predict the low-energy interactions of pions. For example, Weinberg[3] has used current algebra to predict the low-energy behavior of $\pi\pi$ scattering amplitudes. These, and other, current-algebra results are conveniently summarized in effective chiral Lagrangians[4] such as we will utilize later. The low-energy predictions arise from effective Lagrangians with either zero or two space-time derivatives. Higher-order Lagrangians, involving four (or more) derivatives, are also possible but only become important at higher energies.

Recently there has been a revival, started by Witten,[5] of Skyrme's old idea[6] of describing the proton as a topologically stable soliton of the fields in the effective chiral Lagrangians. In this picture the higher-order Lagrangians are essential, as they are required in order to stabilize the soliton. Using a particular choice of higher-order term, Adkins, Nappi, and Witten (ANW) have shown that the properties of this soliton (often labeled the "skyrmion") are reasonably similar to those observed for the proton.[7] In this paper we wish to show that information on $\pi\pi$ scattering uniquely determines the form of the quartic-derivative Lagrangians at tree level, and that the form which results has a soliton with mass consistent (within sizable error bars) with that of the proton. Certainly this is a remarkable connection implied by these chiral theories, as one would in general expect no direct relation between $\pi\pi$ phase shifts and the proton mass. The connection represents a very nontrivial consistency test of the Skyrme models.

Our strategy is to examine $\pi\pi$ phase shifts in the D wave, where the lowest order Lagrangian (i.e., Weinberg's result) does not contribute. At tree level, the leading effect comes solely from the Lagrangians with four derivatives. (We will discuss loops near the end of this note.) The data can empirically determine the coefficients of these Lagrangians, and we shall follow the procedures of Adkins, Nappi, and Witten in order to study the properties of the resulting skyrmion in the chiral limit.

The lowest-order predictions of PCAC are contained in the Lagrangian[4]

$$L_0 = \tfrac{1}{2}F_\pi^2 \operatorname{Tr}(\partial_\mu M \partial^\mu M^\dagger) - \tfrac{1}{2}m_\pi^2 F_\pi^2 (2 - \operatorname{Tr} M), \tag{2}$$

with

$$M = \exp(i\vec{\tau}\cdot\vec{\pi}/F_\pi), \tag{3}$$

and F_π normalized such that the experimental value is $F_\pi = 0.094$ GeV. Expansion of L_0 to fourth order in the pion field yields the Weinberg $\pi\pi$ scattering predictions. A crucial observation, the result of the work of Gasser and Leutwyler,[8] is that there exist only two independent quartic-derivative Lagrangians in the limit that $m_\pi^2 \to 0$. We can choose these to be, e.g.,

$$L_{H0} = (1/32e^2)\operatorname{Tr}\{[(\partial_\mu M)M^\dagger, (\partial_\nu M)M^\dagger]^2\} + (\gamma/8e^2)[\operatorname{Tr}(\partial_\mu M \partial^\mu M^\dagger)]^2. \tag{4}$$

We note that the first term here is that suggested by Skyrme and used by ANW. There exist eight possible forms proportional to m_π^2 [i.e., $(3,3^*)$], but these structures involve only two derivatives and hence cannot affect D-wave scattering. Their contribution to the threshold parameters of the S-wave scattering will be small, and so we shall drop them from further consideration.

The partial-wave expansion for $\pi\pi$ scattering involves the following definitions.[9] The T matrix in a channel with isospin I is

$$T^I = \sum_l (2l+1) P_l(\cos\theta) A_l^I, \tag{5}$$

where θ is the center-of-mass scattering angle and

$$A_l^I = (\sqrt{s}/2q)\exp(i\delta_l^I)\sin\delta_l^I, \quad \mathrm{Re}A_l^I \equiv q^{2l}(a_l^I + b_l^I q^2 + \ldots), \tag{6}$$

with $\sqrt{s}$ being the center-of-mass energy and $q = (s - 4m_\pi^2)^{1/2}/2$. The terminology is such that a_l^I are the scattering lengths and b_l^I are the slopes. A straightforward calculation using the sum of Eqs. (2) and (4), then yields

$$\begin{gathered} a_0^0 = \frac{7m_\pi^2}{32\pi F_\pi^2} + O(m_\pi^4 a_2^0), \quad b_0^0 = \frac{1}{4\pi F_\pi^2} + O(m_\pi^2 a_2^0), \\ a_2^0 = \frac{1}{30\pi e^2 F_\pi^4}(\gamma + \tfrac{1}{2}), \quad b_2^0 = 0, \quad a_2^2 = \frac{1}{30\pi e^2 F_\pi^4}(\gamma - \tfrac{1}{4}), \quad b_2^2 = 0. \end{gathered} \tag{7}$$

Solving for the parameters in the effective Lagrangian, we find

$$F_\pi^2 = \frac{1}{4\pi b_0^0}, \quad \frac{1}{e^2 F_\pi^4} = 40\pi(a_2^0 - a_2^2), \quad \gamma = \frac{1}{4}\left[\frac{a_2^0 + 2a_2^2}{a_2^0 - a_2^2}\right]. \tag{8}$$

When we examine the data later, we shall see that the parameter γ is small so that the simple Skyrme Lagrangian in fact accounts for the major portion of the data. This suggests that it is a good approximation to treat γ as a perturbation to the analysis of ANW, and it is this path which we shall follow. We have arranged the definition of γ to make this route particularly simple. The soliton has the general form $M_0 = \exp[iF(r)\hat{x}\cdot\vec{\tau}]$ and if one allows time-dependent quantum corrections around this solution, $M = A^{-1}(t) M_0 A(t)$, the Lagrangian $L = L_0 + L_{H0}$ (neglecting m_π^2 for simplicity) reduces to

$$L = -M + \lambda\,\mathrm{Tr}[\partial_0 A(t)\partial_0 A^{-1}(t)], \tag{9}$$

where in terms of the dimensionless variable $\tilde{r} = 2eF_\pi r$

$$\begin{gathered} M = \frac{8\pi F_\pi}{e}\int_0^\infty \tilde{r}^2 d\tilde{r}\left\{\frac{1}{8}\left[(F')^2 + \frac{2\sin F}{\tilde{r}^2}\right] + \frac{\sin^2 F}{2\tilde{r}^2}\left[\frac{\sin^2 F}{\tilde{r}^2} + 2(F')^2\right] - \tfrac{1}{2}\gamma\left[(F')^2 + \frac{2\sin^2 F}{\tilde{r}^2}\right]^2\right\}, \\ \lambda = \frac{\pi}{3e^3 F_\pi}\int_0^\infty \tilde{r}^2 d\tilde{r}\,\sin^2 F\left\{\left[1 + 4\left((F')^2 + \frac{\sin^2 F}{\tilde{r}^2}\right)\right] + 8\gamma\left[(F')^2 + \frac{2\sin^2 F}{\tilde{r}^2}\right]\right\}, \end{gathered} \tag{10}$$

with $F' \equiv dF/d\tilde{r}$. To first order in γ we may use the functional form for $F(r)$ determined by a variational approach in Ref. 7. This result is

$$\begin{gathered} M = (73F_\pi/e)f(\gamma), \\ \lambda = (53.3/e^3 F_\pi)g(\gamma) \end{gathered} \tag{11}$$

with

$$\begin{gathered} f(\gamma) = 1 - 0.77\gamma + \ldots, \\ g(\gamma) = 1 + 1.1\gamma + \ldots. \end{gathered} \tag{12}$$

When expressed in terms of the $\pi\pi$ scattering parameters these forms encode the basic connections implicit in the Skyrme models:

$$\begin{gathered} M = 18.4\frac{(a_2^0 - a_2^2)^{1/2}}{(b_0^0)^{3/2}}f(\gamma), \\ \lambda = 134\frac{(a_2^0 - a_2^2)^{3/2}}{(b_0^0)^{5/2}}g(\gamma), \end{gathered} \tag{13}$$

with γ given in Eq. (8). One determines the "proton" mass by projecting out the $I = \frac{1}{2}$, $J = \frac{1}{2}$ collective coordinates,[7] resulting in

$$m_p = M + 3/8\lambda. \tag{14}$$

Finally, we turn to the data in order to assess the numerical significance of this result. We take the value of b_0^0 from a recent compilation of low-energy $\pi\pi$ scattering lengths and slopes,[10]

$$b_0^0 = (13 \pm 2)\ \mathrm{GeV}^{-2}. \tag{15}$$

However, on principle, we feel the best procedure is *not* to use this tabulation for the D waves (although the end result would be very similar; see below). This is because the D-wave scattering lengths are not derived directly from the data but arise from use of dispersion relations involving the Roy equations.[11] For tree-level Lagrangians, we feel it is preferable to use the phase shifts directly. In the D wave the phase shifts are small ($\delta < 15°$) below 1 GeV and unitarity corrections are unimportant. From a direct fit to the phase shifts[12] near 1 GeV we find

$$a_2^0 = (3.4 \pm 1.1)\ \mathrm{GeV}^{-4}, \quad a_2^2 = -(0.5 \pm 0.1)\ \mathrm{GeV}^{-4}, \tag{16}$$

which yields

$$\gamma = 0.16 \pm 0.04, \quad M = 0.69 \pm 0.3\ \mathrm{GeV}, \quad \lambda^{-1} = 0.5 \pm 0.2\ \mathrm{GeV}, \tag{17}$$

and, from Eq. (14)

$$M_p = 0.88 \pm 0.3\ \mathrm{GeV}. \tag{18}$$

The error bars are those of the experimental data only; no attempt has been made to assess the reliability of the theoretical aspects. [We note, however, that had we used the tabulation of scattering lengths we would have obtained consistent values: $a_2^0 = (4.5 \pm 0.8)\ \mathrm{GeV}^{-4}$, $a_2^2 = (0.3 \pm 0.8)\ \mathrm{GeV}^{-4}$, $M_p = 0.81 \pm 0.3$ GeV.] Thus the Skyrme model does in fact pass our consistency test.

This comparison with experiment is not totally without flaw. Surprisingly, the problem lies in the lowest-order Lagrangian, where the standard Weinberg prediction for the $I=0$, $l=0$ scattering lengths and slopes are somewhat smaller than the data. For example, inverting Eq. (7) for the slope or scattering length would imply a value for F_π of $F_\pi = 0.072$–0.077 GeV, which is about 20% low. We note also that the fit value of F_π in Ref. 7 is small by a similar amount. The solution to this problem appears to be the inclusion of one-loop effects[8,13,14] which introduces corrections of $O(m_\pi^2 \ln m_\pi^2)$, alleviating the discrepancy. While we are aware that in a full comparison of chiral Lagrangians with the $\pi\pi$ data one should include such loop corrections, we have purposely chosen not to do so. This is because the effect of loops on the Skyrme solution is as yet unknown. The comparison of $\pi\pi$ scattering at one-loop order with tree-level skyrmions is ill defined, and we feel that it is safest to use the tree-level Lagrangian in both cases. We have attempted to use only those aspects of the data which are least sensitive to such modifications, such as the use of the D waves in $\pi\pi$ scattering, rather than the theoretically more complex S or P waves.

To summarize, we have extracted, from experimental results on $\pi\pi$ scattering, the coefficients of the complete chirally invariant tree-level Lagrangian up to fourth order in the derivatives [Eqs. (2) and (4)]. This has been used to predict the proton mass in a Skyrme model, with results that are successful within the limits of the data. That such a simple relation exists is rather remarkable. The structure of the Lagrangian follows unambiguously from the chiral SU(2) symmetry. However, the study of the soliton solution is somewhat more daring, as it requires that the effective Lagrangians apply to physics at or above 1 GeV. It is about this energy scale that the chiral expansion (in energy) should break down.[2] Put in other words, Lagrangians with even more than four derivatives could become important. Witten has argued that in the large-N_c limit, QCD is equivalent to an effective-field theory of mesons. While the correspondence between the usual quark-model baryon of QCD and the Skyrmion is far from obvious, it is perhaps this equivalence which allows the connection of $\pi\pi$ scattering and the proton mass to be successful.

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[14]We note also that the various contributions of the tree-level Lagrangians to a_0^0 and b_0^0 can help improve the value of F_π. These depend on the (as yet unknown) coefficients of the eight $(\mathbf{3},\mathbf{3}^*)$ Lagrangians, plus the terms of $O(m_\pi^2 a_2^0)$ which were dropped in Eq. (8). We have estimated the latter, and find that they bring the value of F_π to within 10% of experiment, and increase the central value of the proton mass estimate by about 90 MeV.