Topology in Strong Coupling

Nathan Seiberg
The Institute for Advanced Study, Princeton, New Jersey 08540
(Received 14 May 1984)

The lattice CP^{N-1} model at nonzero θ is analyzed by a strong-coupling expansion. At $\theta=\pi$, CP is spontaneously broken and the theory ceases to confine. The θ dependence of several quantities is computed and the correlation length is shown to increase with θ. Most of the results apply not only to the CP^{N-1} model (for all values of N) but to other two-dimensional nonlinear σ models as well.

PACS numbers: 11.15.Ha, 11.10.Lm, 11.30.Er

The analysis of the CP^{N-1} model at large N [1,2] showed that its qualitative behavior as a function of θ is similar to that of the massive Schwinger model at weak coupling.[3] In particular, a cusp in the vacuum energy at $\theta=\pi$ signals the spontaneous breaking of CP. Furthermore, the model does not confine at this point and the quarks are liberated as "half asymptotic states."[3] This behavior is expected to persist for finite values of N including $N=2$ where the model becomes identical to the O(3) σ model. Here we analyze the finite-N model on a Euclidean lattice. Some of the qualitative results which we will find have also been obtained in a Hamiltonian formulation.[4]

The continuum Lagrangian of the model[1,2] is

$$L=N\beta|(\partial_\mu+iA_\mu)\vec{z}|^2, \tag{1}$$

where $\vec{z}$ is an N-component complex unit vector ($\vec{z}^*\cdot\vec{z}=1$). The auxiliary gauge field A_μ may, of course, be eliminated by its equation of motion. A θ term is most easily written in terms of A_μ as

$$i(\theta/2\pi)\epsilon_{\mu\nu}\partial_\mu A_\nu. \tag{2}$$

Several different formulations of the model on the lattice have been given.[5] It is convenient to keep the gauge invariance manifest and to use the lattice action

$$S=-N\beta\textstyle\sum_{x,\mu}[\vec{z}^*(x)U_\mu(x)\vec{z}(x+\hat{\mu})+\text{c.c.}], \tag{3}$$

where $\vec{z}(x)$ are complex unit vectors defined on the sites of the lattice and the gauge field $U_\mu(x)=\exp[i\phi_\mu(x)]$ is an element of U(1) defined on the links. The θ term is somewhat less trivial because, strictly speaking, there is no topology on the lattice. However, following Berg and Luscher,[6] we demand that the topological charge on the lattice has the correct continuum limit and is an integer for every lattice configuration (in a finite volume with periodic boundary conditions). The second requirement guarantees that θ effects will not be seen in weak-coupling perturbation theory. Definitions obeying these requirements were given by Berg and Luscher[6] and Polonyi.[7] Here, motivated by Eq. (2), we will be using a similar, but not identical, definition based on the dummy gauge field $U_\mu(x)$. We define the topological charge density by

$$q(x)=\frac{1}{2\pi i}\ln U_p(x)=\frac{1}{2\pi}\phi_p(x),\quad -\pi\le\phi_p<\pi, \tag{4}$$

where $U_p=\exp(i\phi_p)$ is the oriented product of U_μ around the plaquette. Similar[8] definitions in the context of the Schwinger model were given by Israel and Nappi. Clearly, $q(x)$ has the correct continuum limit. Naively, $Q=\sum_x q(x)=0$ because the link angles cancel in pairs, but because of the branch ambiguity in ϕ_p and the prescription $-\pi\le\phi_p<\pi$, $Q=\sum_x q(x)=$ integer. The complete lattice action with θ is

$$S=-\beta N\textstyle\sum_{x,\mu}[\vec{z}^*(x)U_\mu(x)\vec{z}(x+\hat{\mu})+\text{c.c.}]-i(\theta/2\pi)\sum_x\phi_p(x). \tag{5}$$

The θ dependence of various quantities cannot be seen in a weak-coupling (large-β) expansion. Monte Carlo techniques face the difficulty that the action is not real and $e^{i\theta Q}$ has to be considered as a part of the observable.[9] We therefore turn to a strong-coupling expansion.

In the strong-coupling regime, the configurations are not smooth and all remnant of the topology may be lost. It is therefore surprising that even in this regime, we will find all the qualitative effects that the model is expected to exhibit in the continuum limit.

At zeroth order in the strong-coupling expansion ($\beta=0$) we should consider the partition function

$$Z(\theta,\beta=0)=\exp[-VF(\theta,\beta=0)]=\int_{-\pi}^{\pi}\prod_{x,\mu}d\phi_\mu(x)(2\pi)^{-1}\exp[i(\theta/2\pi)\textstyle\sum_x\phi_p(x)] \tag{6}$$

(V is the number of sites in the lattice). We first consider the system with free-boundary conditions. Picking an axial gauge, we can integrate link after link from the boundary. The value of each integral is independent of the other links and the partition function becomes

$$Z(\theta,\beta=0)=\exp[-VF(\theta,\beta=0)]$$
$$=[(2/\theta)\sin\tfrac{1}{2}\theta]^V. \quad (7)$$

For $|\theta|\leq 2\pi$ we find

$$F(\theta,\beta=0)=-\ln[(2/\theta)\sin\tfrac{1}{2}\theta]. \quad (8)$$

With periodic boundary conditions Q is always an integer and $F(\theta,\beta=0)$ must be a periodic function. In the infinite-volume limit its power-series expansion is independent of the boundary conditions and it is identical to that of Eq. (8). The only effect of the periodic boundary conditions is to make $F(\theta,\beta=0)$ a periodic function. We conclude that

$$F(\theta,\beta=0)=\begin{cases}-\ln[(2/\theta)\sin\tfrac{1}{2}\theta], & |\theta|\leq\pi,\\ F(\theta+2\pi k,\beta=0), & \text{otherwise.}\end{cases} \quad (9)$$

The cusp at $\theta=\pi$ is now obvious.

The expectation value of the topological charge is easily found:

$$\langle q\rangle_{\theta,\beta=0}=\frac{\langle Q\rangle_{\theta,\beta=0}}{V}=i\frac{\partial F(\theta,\beta=0)}{\partial\theta}$$
$$=i(\theta^{-1}-\tfrac{1}{2}\cot\tfrac{1}{2}\theta). \quad (10)$$

The nonzero expectation value of q at $\theta=\pi$ ($\langle q\rangle_{\theta=\pi^\pm,\beta=0}=\mp i/\pi$) shows that CP is spontaneously broken there. Note that the point $\theta=\pi$ does not separate between two different phases; $\theta=\pi^+$ is in the same phase as $\theta=\pi^-$.

There may be some interest in the model at imaginary θ.[10] For large imaginary θ one finds $F(\theta\simeq i\infty,\beta=0)\simeq -i\theta/2$ and $\langle q\rangle_{\theta=i\infty,\beta=0}=\frac{1}{2}$. The topological charge becomes maximal in this limit and there is an "exceptional configuration"[6] on every plaquette.

It is easy to compute the expectation value of the Wilson loop, $\langle W\rangle$, at $\beta=0$. Since for nonzero θ, CP is either explicitly or spontaneously broken, $\langle W\rangle\neq\langle W^\dagger\rangle$, but $\langle W\rangle_\theta=\langle W^\dagger\rangle_{-\theta}$. The computation is similar to that of the free energy and we obtain

$$\langle W\rangle=\left(\frac{\theta}{2\pi-\theta}\right)^A;\quad \langle W^\dagger\rangle=\left(\frac{-\theta}{2\pi+\theta}\right)^A, \quad (11)$$

where A is the area enclosed by the loop. For simplicity we take A to be even and we find two different string tensions,

$$\sigma_1=\ln\left(\frac{2\pi-\theta}{\theta}\right);\quad \sigma_2=\ln\left(\frac{2\pi+\theta}{\theta}\right). \quad (12)$$

Note that $\langle W\rangle_{\theta=\pi}=1$ [$\sigma_1(\pi)=0$] and the theory does not confine at this point. The two different Wilson loops and, correspondingly, the two different string tensions have a simple physical interpretation[3] in a Hamiltonian description. They represent the confining force between a quark (z) and an antiquark (z^*) in two different configurations. The quark is to the right of the antiquark or it is to its left. At $\theta=\pi$, $\sigma_1=0$ and $\sigma_2=\ln 3$. The force in one configuration is still nonzero but the quark and the antiquark in the other configuration do not "feel" any force between them and they are liberated as "half-asymptotic states."[3] The fact that $\sigma_1(\pi)$ vanishes is not accidental here. It is related to the vacuum degeneracy at this point. The functional integral with the loop is identical to the functional integral without the loop but with the replacement $\theta\to\theta-2\pi$ on all the plaquettes surrounded by the loop. It is only at $\theta=\pi$ that the inside of the loop ($\theta=-\pi$) has the same free energy as the outside. The loop surrounds a region of space in one vacuum inside the other vacuum. Since they are degenerate no area law can be found and the string tension vanishes. This interpretation is the Euclidean version of the deconfinement mechanism of Ref. 3.

When β is nonzero but small, we can use standard strong-coupling techniques. We treat the θ dynamics exactly and expand only in β. Clearly, the nonanalyticity at $\theta=\pi$ cannot be removed by small β corrections. It is easy to find

$$F(\theta,\beta)=-\ln\left(\frac{2}{\theta}\sin\frac{\theta}{2}\right)-2N\beta^2+N\beta^4\left(\frac{N}{N+1}-\frac{2\theta^2}{4\pi^2-\theta^2}\right)+O(\beta^6). \quad (13)$$

The Wilson loop cannot exhibit an area law for nonzero β; the dynamical quarks screen it. Indeed, a straightforward computation yields

$$\langle W\rangle_{\theta\neq\pi}\sim\exp\{-P[\ln\beta^{-1}+O(\beta^2)]\};\quad \langle W^\dagger\rangle_{\theta\neq-\pi}\sim\exp\{-P[\ln\beta^{-1}+O(\beta^2)]\}, \quad (14)$$

$$\langle W\rangle_{\theta=\pi}\sim\exp\{-P[\tfrac{16}{9}N\beta^6+O(\beta^8)]\}, \quad (15)$$

where P is the perimeter of the loop.

Another object of interest is the mass gap $m(\theta,\beta)$. It can be found by computing the correlation function

$$\sum_x \langle z_a^* z^b(x,0) z_c^* z^d(0,L)\rangle = f(\theta,\beta,L)\delta_a^d\delta_c^b \quad \text{for} \quad a\neq b. \tag{16}$$

In the strong-coupling limit

$$f(\theta,\beta,L)=\frac{1}{N(N+1)}\left(\frac{N\beta^2}{N+1}\right)^L\left\{1+\frac{2N\beta^2}{N+1}+L\frac{2\beta^2 N}{(N+1)(N+2)}\right.$$
$$\left.+4\beta^2\sum_{k=1}^{L}(L-k+1)\left(\frac{N+1}{N}\right)^{k-1}\left[\left(\frac{\theta}{2\pi-\theta}\right)^k+\left(\frac{-\theta}{2\pi+\theta}\right)^k\right]+O(\beta^4)\right\} \tag{17}$$

Hence

$$m(\theta,\beta)=-\lim_{L\to\infty}L^{-1}f(\theta,\beta,L)$$
$$=\begin{cases}\ln\dfrac{N+1}{N\beta^2}-2\beta^2\dfrac{N}{(N+1)(N+2)}-8\beta^2\dfrac{N}{2N+1}\dfrac{\theta^2}{(\theta_c^2-\theta^2)}+O(\beta^4), & |\theta|<\theta_c=\pi\dfrac{2N}{2N+1},\\ \ln\dfrac{2\pi-\theta}{\theta\beta^2}+O(\beta^2), \quad \theta_c<|\theta|<\pi. & \end{cases} \tag{18}$$

The apparent singularity in the $O(\beta^2)$ term at $m(\theta_c,\beta)$ is an artifact of the expansion in β. A more detailed analysis at $\theta=\theta_c$ leads to a contribution $O(|\beta|)$—

$$m(\theta=\theta_c,\beta)$$
$$=\ln\left(\frac{N+1}{N\beta^2}\right)-2|\beta|\left(\frac{N}{N+1}\right)^{1/2}+O(\beta^2)$$

and $m(\theta,\beta)$ is smooth (there is no level crossing) at $\theta=\theta_c$. At this point, the repulsive contact interaction among the z particles is equal to the string tension, and as a result, $m(\theta,\beta)$ exhibits a crossover. The z-z^* contact term is not universal and hence, this crossover should, presumably, disappear in the continuum limit. Similar phenomena occur near $\theta=\pi$. At this point the quarks are liberated and $m_z=m_{z^*}=\ln\beta^{-1}+O(|\beta|)$. It is important to note that for fixed β, $m(\theta,\beta)$ decreases with θ, or equivalently, the correlation length $\xi=1/m$ increases with θ. This fact agrees with theoretical expectations[11] and numerical simulations.[9]

In conclusion, θ has been introduced into a lattice version of the CP^{N-1} model in a simple manner that enables a straightforward strong-coupling expansion. θ was treated exactly and the only small expansion parameter was β. We observed a cusp in the free energy at $\theta=\pi$, spontaneous breaking of CP, and quark liberation as "half asymptotic states" at that point, and the fact that the correlation length increases with θ. These effects are expected (although they have not yet been proven) in the continuum limit. The fact that we found them in the strong-coupling region may serve as another indication that these expectations are reasonable.

Clearly, we cannot prove that these phenomena persist all the way to the continuum limit. We can only point out that since the strong-coupling expansion has a finite radius of convergence[12] [which can easily be shown by comparing the diagrams with those of the O($2N$) model], the structure at $\theta=\pi$ persists for at least a finite range of β. Moreover, the absence of a phase transition at $\theta=0$ for any β makes it plausible that qualitative features of the strong-coupling physics are present in the continuum. Furthermore, the large N and the strong-coupling limits of this lattice action are known not to commute.[5] Yet, the large-N limit at the continuum and the finite-N strong-coupling limit exhibit similar effects at $\theta=\pi$. It seems, therefore, likely that these phenomena occur for all values of N at all couplings including $\beta\to\infty$—the continuum limit.

So far we considered the CP^{N-1} model. However, the method is applicable for any two-dimensional model for which the topological term can be written in terms of a (possibly dummy) U(1) gauge field. For instance, a nonlinear σ model based on the manifold U($2N$)/U(N)×U(N)[13] can be analyzed in a similar way. The $\beta=0$ results are clearly independent of the details of the model and are, therefore, identical to the results found here. In particular, at least for $\beta=0$, the $N\to 0$ limit of this model, which may be relevant to the quantized Hall effect,[13] is smooth and has all the features discussed here.

I thank T. Banks, R. Dashen, M. Dine and E. Witten for discussions. This work was supported in part by the Department of Energy under Contract No. DOE-AC02-76ER02220.

[1]D'Adda, P. Di Vecchia, and M. Luscher, Nucl. Phys. **B146**, 63 (1978).

[2]E. Witten, Nucl. Phys. **B149**, 285 (1979).

[3]S. Coleman, Ann. Phys. (N.Y.) **101**, 239 (1976), and references therein.

[4]F. D. M. Haldane, Phys. Rev. Lett. **50**, 1153 (1983), and references therein; E. Witten, private communication; T. Banks, private communication.

[5]M. Stone, Nucl. Phys. **B152**, 97 (1979); E. Rabinovici and S. Samuel, Phys. Lett. **101B**, 323 (1981); P. Di Vecchia, A. Holtkamp, R. Musto, F. Nicodemi, and R. Pettorino, Nucl. Phys. **B190** [FS3], 719 (1981); I. Affleck and H. Levine, Nucl. Phys. **B195**, 493 (1982); P. Di Vecchia, R. Musto, F. Nicodemi, R. Pettorino, and P. Rossi, CERN Report No. TH. 3753, 1983 (to be published).

[6]B. Berg and M. Luscher, Nucl. Phys. **B190** [FS3], 412 (1981).

[7]J. Polonyi, Illinois Report No. ILL-TH-83-3, 1983 (to be published).

[8]R. B. Israel and C. R. Nappi, Commun. Math. Phys. **64**, 177 (1979); D. Brydges, J. Frohlich, and E. Seiler, Ann. Phys. (N.Y.) **121**, 227 (1979).

[9]G. Bhanot, E. Rabinovici, N. Seiberg, and P. Woit, Nucl. Phys. **B230** [FS10], 291 (1984); G. Bhanot, R. Dashen, N. Seiberg, and H. Levine, Phys. Rev. Lett. **53**, 519 (1984).

[10]G.'t Hooft, in Proceedings of the European Physical Society Conference on High Energy Physics, Lisbon, 1981 (unpublished); G. Bhanot and F. David, to be published.

[11]C. Callan, R. Dashen, and D. Gross, Phys. Rev. D **20**, 3279 (1978).

[12]K. Osterwalder and E. Seiler, Ann. Phys. (N.Y.) **110**, 440 (1978).

[13]H. Levine, S. B. Libby and A. M. M. Pruisken, Phys. Rev. Lett. **51**, 1915 (1983), and to be published.