Self-Focusing–Induced Optical Turbulence

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(Received 21 March 1984)

Transverse outputs of an optical resonator are predicted to undergo transitions to turbulence via three well-established scenarios. The instabilities are induced by self-focusing nonlinearities. A bifurcation involving period doubling of invariant circles in a parameter region where attractors coexist is identified.

PACS numbers: 42.65.Jx, 05.40.+j

Spatiotemporal effects play an important role in the transition to turbulence in low–aspect-ratio fluids.[1] In both Rayleigh-Bénard convection and Taylor-Couette flow, the onset of a new temporal motion (e.g., periodic to quasiperiodic motion) is accompanied by the sudden appearance of small periodic spatial scales in the fluid. A Fourier mode expansion of the Navier-Stokes equations describing two-dimensional fluid flow leads at the lowest level of truncation to the well-known Lorenz equations. These latter equations undergo transitions to chaos and Haken[2] pointed out their close similarity to the plane-wave single-mode laser equations. However, it is not clear to what extent these severely truncated models mimic the true physical situation. While these models can exhibit complex bifurcation structure, the nature of the transition to turbulence becomes sensitive to the level of truncation.[3]

In the present work I address the problem of spatiotemporal structures in a nonlinear optical scenario. My model is an optical bistable ring resonator [Fig. 1(a)] with an incident cw laser beam having a Gaussian spatial profile (one transverse dimension). The nonlinear medium consists of saturable two-level atoms and I assume that it is operating in the purely dispersive limit. A new series of transitions to turbulence induced solely by self-focusing effect is identified; under self-defocusing conditions or in the plane-wave model the corresponding motion is at most periodic (period two) over the same parameter range. Three typical transition sequences observed in the above fluid experiments are also predicted to occur in this optical system. They are (1) period doubling to chaos, (2) periodic-quasiperiodic breakdown of a two-torus to a chaotic attractor (Ruelle-Takens scenario), and (3) periodic-intermittent chaos (Pomeau scenario). At the onset of quasiperiodic motion, a profound change occurs in the transverse spatial profile; oscillatory spatial rings appear [Figs. 1(b) and 1(c)] that are predominantly solitary waves of the nonlinear wave equation [Eq. (1) below]. Exceptions to the above transition sequences usually require the coexistence of attractors in phase space (a situation encountered in the optical model). Observation of three incommensurate frequencies in the power spectra of time series of radial fluid-velocity profiles (evidence for motion on a three-torus in the phase space) is typical of such behavior. This latter behavior has recently been predicted to occur in an optical scenario.[4] I predict that a bifurcation,[5] involving period doubling of invariant circles, can occur in a region of parameter space where attractors coexist. My study provides an example of how one may extract pictures of few-dimensional bifurcating attractors from an infinite-dimensional dissipative dynamical system.

The idea is to look for asymptotic states of the output field as either the input Gaussian peak amplitude $a(0)$ or the "effective propagation length" $p(=\alpha_0 L_1/\Delta)$ of the nonlinear medium is varied. As discussed by Moloney and co-workers,[6] the

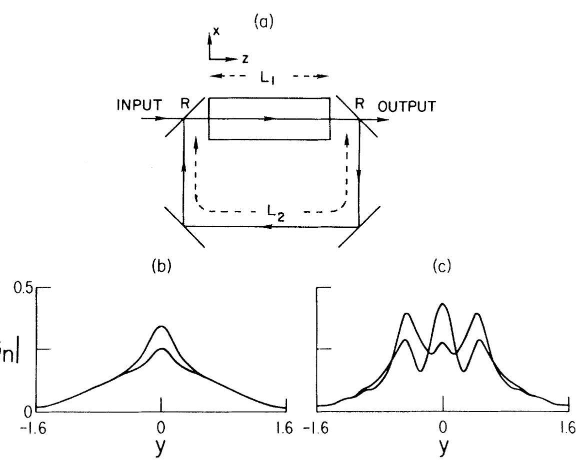

FIG. 1. (a) Schematic of a unidirectional passive ring resonator containing a nonlinear (two-level atom) saturable medium of length L_1. (b) Period-two oscillation near the center of the focused beam. (c) Spatial rings appear at the point of bifurcation from periodic to quasiperiodic motion. Fresnel number $F=5$ and $L=0.4$ in this and following figures.

dynamical behavior of the electromagnetic field in the resonator is described by the following nonlinear evolution equation (in one transverse spatial dimension):

$$2i\frac{\partial}{\partial\zeta}G_n+\frac{\partial^2}{\partial y^2}G_n-\frac{G_n}{1+2|G_n|^2}=0, \qquad (1)$$

and the resonator boundary conditions

$$G_n(y,0)=a(y)+Re^{ikL}G_{n-1}(y,p). \qquad (2)$$

Equations (1) and (2) together constitute an infinite-dimensional complex map in discrete time where the index n counts the number of circuits of the field around the resonator. $G_n(y,\zeta)$ is the normalized intracavity field amplitude where y and ζ refer respectively to the coordinates in the transverse and propagation directions. These equations are written in a convenient nondimensional form through the following scaling:

$$p=\frac{\alpha_0 L_1}{\Delta},\quad f=\frac{4_\pi F_p}{\ln 2},\quad \zeta=\frac{\alpha_0 z}{\Delta},\quad y=(fx)^{-1/2},$$

$$a(y)=\left(\frac{T}{2\Delta^2}\right)^{1/2}A\left(\frac{y}{\sqrt{f}},L_1,\frac{\zeta}{p}\right),$$

$$G_n(y,\zeta)=(2\Delta)^{-1/2}B_n(y/\sqrt{f},L_1,\zeta/p),$$

where α_0 is the linear absorption coefficient characterizing the medium, L_1 is the medium length $[L=L_1+L_2$; see Fig. 1(a)], $T=1-R$ is the mirror intensity transmission coefficient, Δ is a normalized frequency detuning, and F, the Fresnel number, measures the importance of diffraction in the linear problem. These and the other parameters are defined in detail in Ref. 6. The calculational procedure is first to solve the nonlinear evolution equation (1) over the "effective" nonlinear medium length p with initial data $a(y)$, then to substitute into the right-hand side of Eq. (2), and to determine new initial data for Eq. (1). The procedure is repeated until some asymptotic state is reached.

This system is known to exhibit hysteresis and dynamic instabilities[7] and it has recently been shown numerically that, when a high-Fresnel-number self-focusing ($p>0$) beam switches to a high-transmission state of the bistable system, spatial ring structures evolve across the turned-on beam.[6(b)] Subsequently, it was established that these spatial rings were solitary waves of the nonlinear evolution equation (1) and fixed points of the above infinite-dimensional map.[6(a)] In the discussion that follows, we confine our attention to instabilities in a region where no hysteresis occurs; physically this corresponds to a situation where, as a function of increasing laser-input peak amplitude $a(0)$, the field-induced nonlinear index change in the medium tunes the system away from a nearby cavity-transmission peak. Complications arising from interaction with nearby unstable (middle) or stable (upper) branches of a hysteresis loop are thereby avoided.[8] If we ignore the second-derivative term in Eq. (1) the plane-wave model of dispersive optical bistability follows immediately from Eqs. (1) and (2). This latter model has been studied extensively in the literature.[9]

The transverse outputs from the ring resonator may be recorded by detecting the total integrated output intensity or, alternatively, by using a small aperture to record, say, the center beam intensity. The latter technique, which will be assumed here, is similar in spirit to the use of a Doppler probe to record radial fluid-velocity profiles at a fixed spatial location in a fluid.[1] Figure 2 summarizes the type of bifurcation sequences that arise as we vary either $a(0)$ (the peak input Gaussian amplitude) or p (the "effective" nonlinear-medium length). Typical sequences involve a period-doubling bifurcation from a stable fixed point to a periodic orbit, followed by a Hopf bifurcation to an invariant circle. As noted earlier, both the plane-wave model and the present model under self-defocusing conditions exhibit either a stable fixed point or a simple periodic motion (period two) over the present parameter range of interest. Motion on the invariant circle may be quasiperiodic or may lock to some periodic orbit. If we view the map as suspended in a flow of one higher dimension, this latter motion may be interpreted as lying on an invariant two-torus (T^2). Breakdown of the motion on the invariant circle can either be direct to a chaotic attractor[10] (a typical scenario in Fig. 2) or involve a period-doubling cascade starting from a frequency-locked periodic orbit on the invariant circle.[7] Along the line $p=8$ in Fig. 2 we observe a direct transition from a period-two orbit to an intermittent chaotic attractor. Of particular significance is the coexistence of attractors over wide ranges of parameter space, a situation also encountered in the plane-wave map. Iooss and Langford[11] conjecture that interaction between coexisting attractors can be responsible for higher codimension bifurcations in many-dimensional physical systems.

Figure 3 provides a geometric picture of the attractors embedded in a three-dimensional phase space as we progress through the bifurcation sequence on the line $p=8$, $0.27<|a(0)|^2<0.32$ in Fig. 2. Discrete time records of the output beam-center amplitude are used to construct the embed-

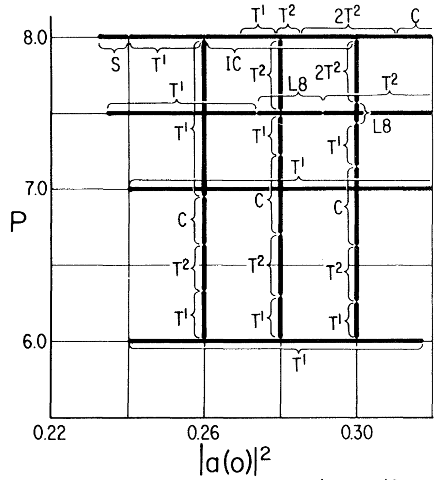

FIG. 2. Bifurcation diagram in $(p, |a(0)|^2)$ parameter space. Typical bifurcation sequences involve S (stable) $\rightarrow T^1$ (one-torus for flow or period-2 orbit) $\rightarrow T^2$ (two-torus or invariant circle) $\rightarrow$ C (chaos) $\rightarrow T^1$. To the left, on the line $p=8$, we have S $\rightarrow T^1 \rightarrow$ IC (intermittant chaos). Coexisting with the intermittant chaotic attractor is the sequence $T^1 \rightarrow T^2 \rightarrow 2T^2$ (doubled torus or double circle) $\rightarrow$ C (chaos) shown in detail in Fig. 3. Along the vertical line $|a(0)|^2=0.3$, $p \geqslant 7.5$ bifurcation occurs from a period-2 orbit directly to a frequency-locked periodic orbit (L8).

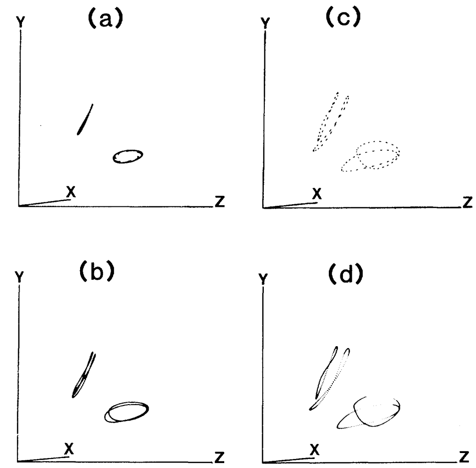

FIG. 3. Bifurcating attractors along the line $p=8$ embedded in a three-dimensional phase space are reconstructed from discrete time series of the beam-center amplitude, (a) $|a(0)|^2=0.285$, (b) $|a(0)|^2=0.2875$, (c) $|a(0)|^2=0.29$, and (d) $|a(0)|^2=0.295$. Doubling of the invariant circles occurs on going from (a) to (b).

ded attractors. These pictures convey much more information than time series or power spectra; for example, we can easily distinguish between a high-period orbit [Fig. 3(c)] and quasiperiodic motion [Fig. 3(b)] on the invariant circles. The bifurcation sequence starts at a simple period-two orbit (not shown). A Hopf bifurcation leads to quasiperiodic motion on the two invariant circles [Fig. 3(a)]. At this point in the Ruelle-Takens scenario, one would expect the invariant circles to break down to a chaotic attractor either directly[5,10] or via a frequency locking on the circles followed by a period-doubling cascade of the locked output.[7] Instead, the Hopf bifurcation is followed closely by a period doubling of the invariant circles. From a bifurcation-theory point of view, this latter behavior requires that a complex-conjugate pair of eigenvalues {of the linearization of the map [Eqs. (1) and (2)]} cross the unit circle at an irrational angle (Hopf bifurcation) followed closely in parameter space by a single eigenvalue crossing the unit circle along the real negative axis (flip or period-doubling bifurcation). The doubled circles break down to a chaotic attractor by developing kinks and passing through a complicated series of higher-period frequency lockings (Fig. 4). Along the vertical line $|a(0)|^2=0.3$ $(7 < p \leqslant 8)$ the behavior is qualitatively similar except that the eigenvalues cross the unit circle at the rational angle $\alpha = 2\pi m/q$ $(m=1, q=4)$ leading immediately to a frequency-locked output. Details of this transition and the other behaviors depicted in Fig. 2 will be discussed elsewhere.

In conclusion, I have identified new routes to optical turbulence in a bistable ring resonator. The instabilities are induced by self-focusing nonlineari-

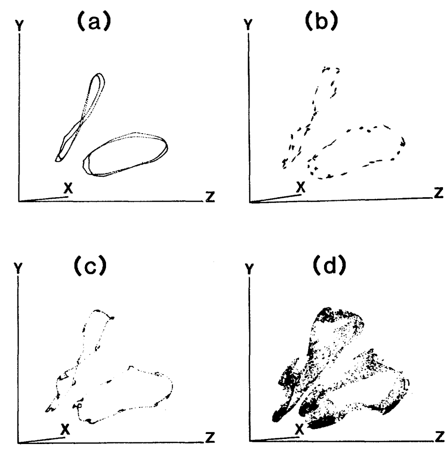

FIG. 4. Breakdown of the invariant circles along the line $p=8$ to a chaotic attractor involves the development of kinks [(a) $|a(0)|^2=0.31$], high-period frequency lockings [(b) $|a(0)|^2=0.315$], and complicated fractal curves [(c) $|a(0)|^2=0.32$]. The final chaotic attractor consists of two pieces reflecting the original underlying period-2 orbit.

ties and the optical analogs of the small spatial scales observed in fluid experiments appear to be transverse solitary waves of the nonlinear wave equation (1). Three-dimensional phase-space portraits have been reconstructed from discrete time records of the output beam-center amplitude showing that although our system is formally infinite dimensional, asymptotic motion lies on few-dimensional attractors.

The author benefitted from stimulating conversations with Chris Jones, Hermann Flascka, and Philip Holmes. This work was supported by grants from the National Science Foundation and the Air Force Office of Scientific Research.

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