Parity Conservation in Quantum Chromodynamics

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We show that in parity-conserving vectorlike theories such as QCD, parity conservation is not spontaneously broken.

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Recently, rigorous inequalities have been used to obtain information about which symmetries are spontaneously broken in QCD. Weingarten[1] proved an inequality which—given anomaly constraints[2] and an assumption of confinement—proves that the axial-vector (chiral) symmetries of quarks must be spontaneously broken. We have elsewhere proved,[3] with mild technical assumptions, that the vector symmetries, such as isospin and baryon number, cannot be spontaneously broken. These results apply, in fact, not just to QCD but to arbitrary parity-conserving vectorlike theories of fermions and gauge mesons[4] (at $\theta=0$—a restriction that is always assumed in what follows). In this paper we will consider from a similar point of view the discrete symmetries—C, P, and T. The argument we will give is an extension of one we used previously in discussing gauge theories in 2+1 dimensions.[5]

The CPT theorem states that the product CPT of the discrete symmetries is unbroken. (The most easily proved statement is that arbitrary local relativistic Lagrangians conserve CPT, but the CPT theorem properly states in addition that the vacuum and all observables are CPT invariant.[6]) In this paper we will show that P is not spontaneously broken in parity-conserving, vectorlike theories such as QCD. Here P is the standard transformation $[A_0(\vec{x},t),\ A_i(\vec{x},t)] \to [A_0(-\vec{x},t),\ -A_i(-\vec{x},t)]$. We do not know of a general argument excluding C or T breakdown in QCD. However, since P and CPT are both unbroken, if either C or T is unbroken so is the other.

What we will actually prove is that the vacuum expectation value of an arbitrary Hermitian, P-nonconserving observable X is zero in theories such as QCD. The argument that follows is most natural if X is a local operator constructed from Bose fields only. There are an infinite number of such P-nonconserving operators [such as $\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}F_{\mu\nu}F_{\alpha\beta}$, $\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}D_\sigma F_{\mu\nu}D_\sigma F_{\alpha\beta}$, $(\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}F_{\mu\nu}F_{\alpha\beta})^3$, etc.], and it is implausible that they all have zero vacuum expectation value if P is spontaneously broken. However, our argument does not really require X to be a local operator and can be carried out even if X involves Fermi fields.[7]

Let $\mathscr{L}$ be the Lagrangian of the theory under consideration, and let $\hat{\mathscr{L}}(\lambda)$ be the generalized Lagrangian $\mathscr{L}-\lambda X$. For real λ, $\mathscr{L}$ is Hermitian, since we assume X Hermitian. Let $E(\lambda)$ be the vacuum energy of the theory with Lagrangian $\hat{\mathscr{L}}(\lambda)$. To lowest order in λ, $E(\lambda)=E(0)+\lambda\langle X\rangle$, where $\langle X\rangle$ is the vacuum expectation value of X at $\lambda=0$. If $\langle X\rangle$ is not zero, it can have either sign (since X is odd under parity), so that regardless of the sign of λ, the theory can choose a vacuum state in which $\lambda\langle X\rangle$ is negative. The statement that $\langle X\rangle \neq 0$ would therefore imply that for small, nonzero λ, $E(\lambda)<E(0)$.

To show that this is impossible, we will consider the Euclidean path-integral expression for $E(\lambda)$. We must first discuss what happens to parity-nonconserving operators under Wick rotation. The Hermitian operator X is real in Minkowski space. To be Lorentz invariant, X is constructed from the gauge field A_μ^a, the metric tensor $g_{\mu\nu}$, and the antisymmetric tensor $\epsilon_{\mu\nu\alpha\beta}$. Although A_μ^a and $g_{\mu\nu}$ remain real in Euclidean space, $\epsilon_{\mu\nu\alpha\beta}$ picks up a factor of i under Wick rotation. A Lorentz-invariant but parity-nonconserving operator X must be proportional to an odd power of $\epsilon_{\mu\nu\alpha\beta}$, and so X picks up a factor of i in Wick rotation.

In a Euclidean volume V, the path-integral formula for the ground-state energy is

$$e^{-VE(\lambda)} = \int dA_\mu^a\, d\psi\, d\bar{\psi}\, \exp[-\int d^4x\,(\mathscr{L}+i\lambda X)], \quad (1)$$

where we have exhibited the factor of i that arises from Wick rotation. When we integrate out the fermions, this becomes

$$e^{-VE(\lambda)} = \int dA_\mu^a \det(\not{D}+M)\exp[-(4g^2)^{-1}\int d^4x\, \mathrm{Tr}F_{\mu\nu}F_{\mu\nu}]\exp(i\lambda\int d^4x\, X), \tag{2}$$

where M is the fermion mass matrix. In vectorlike theories, the fermion integral $\det(\not{D}+M)$ is positive,[3] so that the integrand in (2) is positive except for the phase factor $\exp(i\lambda\int d^4x\, X)$. Inclusion of this phase factor can only make the integral less, and so $E(\lambda)$ has its minimum at $\lambda=0$. Therefore, the vacuum expectation value of X is zero at $\lambda=0$, and parity is not spontaneously broken in theories such as QCD.

Incidentally, the same argument applied to the θ dependence of the QCD vacuum energy

$$e^{-VE(\theta)} = \int dA_\mu^a \det(\not{D}+M)\exp[-(4g^2)^{-1}\int d^4x\, \mathrm{Tr}F_{\mu\nu}F_{\mu\nu}]\exp\left(\frac{i\theta}{16\pi^2}\int d^2x\, \epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}F_{\mu\nu}F_{\alpha\beta}\right) \tag{3}$$

shows that the minimum energy is at $\theta=0$. This is essential for the viability of the axion approach to the strong CP problem, in which the vacuum angle is a dynamical variable and relaxes to the value that minimizes the energy.[8] If the gauge field A_μ in QCD is regarded as a 3×3 anti-Hermitian matrix, then the standard time-reversal transformation is $[A_0(\vec{x},t),\ A_i(\vec{x},t)] \to [-A_0^*(\vec{x},-t),\ A_i^*(\vec{x},t)]$; here A_μ^* is the complex conjugate (not adjoint) of A. It is possible to find Hermitian operators odd under this transformation and real in Euclidean space, so that our argument cannot be extended to exclude spontaneous breakdown of T conservation. The standard C transformation is $A_\mu(\vec{x},t) \to A_\mu^*(\vec{x},t)$; again Hermitian, C-nonconserving operators can be real in Euclidean space, and so our argument does not apply to C conservation. The standard CT transformation is $[A_0(\vec{x},t),\ A_i(\vec{x},t)] \to [-A_0(\vec{x},t),\ A_i(\vec{x},t)]$. Hermitian operators odd under this transformation are indeed always imaginary in Euclidean space, so that our argument does apply to CT. The fact that the argument applies to P and CT is simply one aspect of the CPT theorem.

Perhaps it is worthwhile to point out that at $\theta=\pi$, parity is spontaneously broken in a certain range of quark bare masses.[9,10] Therefore, positivity of the Euclidean measure is necessary as well as sufficient for proving that parity is not spontaneously broken in QCD.

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[2]G. 't Hooft, in *Recent Developments in Gauge Theories*, edited by G. 't Hooft *et al.* (Plenum, New York, 1980).

[3]C. Vafa and E. Witten, Nucl. Phys. **B234**, 173 (1984).

[4]Scalars can be included in certain limited ways (Ref. 3).

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[6]R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics, and All That* (Benjamin, New York, 1964).

[7]For instance, let $X=\bar{q}i\gamma_5 q(x)$. When we integrate out the Fermi fields, X becomes $X=\mathrm{Tr}\,i\gamma_5 S_A(x,x)$, where S_A is the fermion propagator in the background A field. Our argument can be applied to X, which is a functional of the A field only.

[8]This argument needs to be specified more carefully. There are two values of θ, $\theta=0$ and $\theta=\pi$, at which CP is conserved. Which is realized in nature? The state that minimizes the energy has the property that $\langle m_u\bar{u}u\rangle$, $\langle m_d\bar{d}d\rangle$, and $\langle m_s\bar{s}s\rangle$ are all negative, as this makes the shift in vacuum energy due to quark masses $\langle m_u\bar{u}u + m_d\bar{d}d + m_s\bar{s}s\rangle$ as negative as possible. This is the state that is realized in nature, because the Gell-Mann–Oakes–Renner analysis of the pseudoscalar masses [M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. **175**, 2195 (1968)] shows that $\langle m_u\bar{u}u\rangle$, $\langle m_d\bar{d}d\rangle$, and $\langle m_s\bar{s}s\rangle$ are all negative in nature. If θ were shifted by π, $\langle m_u\bar{u}u\rangle$ would have opposite sign from the others [R. J. Crewther, P. diVecchia, G. Venezidno, and E. Witten, Phys. Lett. **88B**, 123 (1979)], contradicting what emerges from the Gell-Mann–Oakes–Renner analysis. The observation that the state realized in nature is the state of minimum energy is all that is needed to make the axion theory viable. As to whether this minimum energy state is $\theta=0$ or $\theta=\pi$, this depends on certain conventions. If we choose phase conventions so that even in the instanton sector $\det(\not{D}+M)$ is always positive, then the meaning of θ is fully specified and the argument in the text shows that the minimum energy is at $\theta=0$.

[9]Crewther, diVecchia, Veneziano, and Witten, Ref. 8.

[10]R. Dashen, Phys. Rev. D **3**, 1879 (1971).