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Critical Properties of an Elastic Fractal

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Some solvable fractal models of the percolating backbone are used to investigate the critical behavior of a random, d-dimensional, isotropic, elastic medium. The critical exponent T for the elastic moduli is found to be appreciably greater than the conductivity exponent t, and the ratio of bulk to shear modulus is found to have the universal value $4/d$. A comparison with the effective-medium and the Clausius-Mossotti–type approximations leads to the conjecture that the result $4/d$ is in fact exact.

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An inhomogeneous random mixture of a solid and a fluid will only exhibit solid properties (e.g., a nonvanishing shear modulus μ) if the solid component forms a percolating cluster, as in the sol-gel transition of polymer solutions. If instead of the fluid component we have a vacuum, the bulk modulus κ will also vanish below the percolation threshold. In this Letter we investigate the critical behavior of μ and κ by invoking some solvable fractal models for the backbone of the percolating cluster at threshold. Rather surprisingly, we find that the critical behavior of the elastic moduli μ and κ differs from that of the electrical conductivity σ in the same models. We also find that in the two-dimensional (2D) case, the ratio κ/μ has the universal value 2. We present and discuss the conjecture that the universal value of κ/μ in real d-dimensional percolating systems near the threshold is $4/d$.

The fractal object that we consider is the 2D Sierpinski gasket[1] and its generalizations to arbitrary integral dimensionalities d. This fractal was first proposed as a model for the backbone by Gefen *et al.*[2] Even though it is clearly not a very good model if judged by the numerical values that it yields for the conductivity critical exponent t [$\sigma_e \propto (p-p_c)^t$ where p is the volume fraction of the conducting component and p_c is the percolation threshold], it has the advantage of being exactly solvable. By using this same model to evaluate the critical exponent T of the elastic moduli,

$$\kappa_e, \mu_e \propto (p-p_c)^T, \tag{1}$$

we can thus test the hypothesis, put forward some years ago by de Gennes,[3] that $T=t$.

Consider first the 2D gasket shown in Fig. 1(a), of size $L=2^l$ for $l=2$. Each bond represents a simple spring with a spring constant k_0. Macroscopic external forces are applied only at the three external vertices. In order to construct a network with equivalent macroscopic elastic properties but a smaller number of unit (i.e., $L=1$) cells, we try to replace every $L=2$ piece of the gasket by an equivalent $L=1$ piece. Because every piece is connected to the rest of the spring network by its three external vertices, one only has to consider the dif-

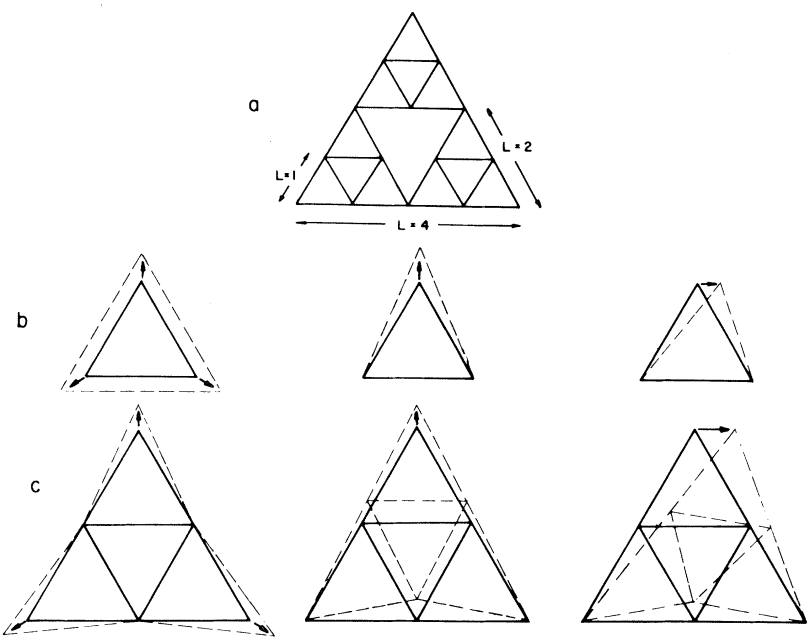

FIG. 1. (a) Sierpinski gasket in 2D with $L=4$ unit cells to a side. Each bond corresponds to a simple spring with spring constant k_0. (b),(c) Three possible deformations of the $L=1$ and $L=2$ gaskets, respectively. Note that to order δ_L, only the external bonds change their lengths.

ferent ways of deforming those vertices. Three such deformations are shown in Fig. 1(b) for an $L=1$ gasket and in Fig. 1(c) for an $L=2$ gasket. Neglecting deformations that are of higher order in the overall deformation $\delta L/L$, we find that only external bonds in Fig. 1(c) actually change their length. In all cases the number of bonds affected in the $L=2$ gasket is twice the number affected in the $L=1$ gasket, and each of the affected bonds changes by the same amount. These conclusions are easily seen to be valid for Sierpinski gaskets in any number of dimensions. This enables us to compare the elastic deformation energies of the $l=0$ $(L=1)$ and $l=1$ $(L=2)$ gaskets $E_0=Ak_0\times\delta L^2$ and $E_1=2Ak_0(\delta L/2)^2$, respectively, where the constant A depends on the character of the deformation. Thus, we can replace the $L=2$ gasket by a simple triangle of springs of size $L=2$ if we rescale the spring constant to be $k_1=k_0/2$.

The elastic moduli for a size-$L=2^l$ gasket made of elementary springs are thus the same as the elastic moduli of a simple size-L triangle with a spring constant $k_l=2^{-l}k_0=k_0/L$. Thus we may write, for any elastic modulus C,

$$C_L\propto k_l/L^{d-2}=k_0/L^{d-1}. \tag{2}$$

The critical behavior of the macroscopic modulus $C(p)$ is now easily found, following Gefen *et al.*,[2] by identifying $C(p)$ with C_L for $L=\xi$, where ξ is the percolation correlation length $\xi\propto(p-p_c)^{-\nu}$. Thus we find

$$C(p)\propto(p-p_c)^T, \quad T/\nu=d-1. \tag{3}$$

In contrast, when the same fractal model of the percolating backbone was used for the conductivity problem, the result obtained was[2]

$$\sigma(p)\propto(p-p_c)^t, \quad \frac{t}{\nu}=d-2+\log_2\frac{d+3}{d+1}. \tag{4}$$

Comparing these results for $d=2,3$ we find

$$\begin{aligned} &t/\nu=0.737,1.585 \text{ for } d=2,3,\\ &T/\nu=1,2 \text{ for } d=2,3. \end{aligned} \tag{5}$$

Apart from the rather artificial geometry of the Sierpinski gasket, the main flaw of the model discussed above seems to be the fact that there is only one "microscopic" elastic constant k_0. Thus it should come as no surprise that the bulk modulus κ and the shear modulus μ that can be defined for this model have a fixed ratio κ/μ. This ratio can be found by calculating the elastic energies associated with two different distortions of the unit cell of the gasket. Thus, if the unit cell is compressed isotropically by contracting each bond of length L by an equal amount δL, the elastic energy is

$$\tfrac{1}{2}k_0[d(d+1)/2]\delta L^2, \tag{6}$$

where the factor $d(d+1)/2$ is the number of bonds in the unit cell, which is a d-dimensional simplex. This can be equated to the expression for the energy in terms of the Lamé constants λ and μ for a

fractional volume change of $d\delta L/L$:

$$\tfrac{1}{2}[\lambda+(2/d)\mu](d\delta L/L)^2, \tag{7}$$

which yields

$$\kappa=\lambda+(2/d)\mu=(k_0L^2/2)(d+1)/d. \tag{8}$$

Similar considerations can be made for the case where only a $(d-1)$-dimensional simplex that is part of the d-dimensional simplex is isotropically compressed, but there are no distortions in the perpendicular direction. (In 2D this would mean compressing a basis of the unit triangle, and leaving the perpendicular height unchanged.) We thus find the following equivalent expressions for the elastic energy[4]:

$$\frac{k_0\delta L^2}{2}\left[\frac{d(d-1)}{2}+\frac{(d-1)^2}{4d}\right]=\frac{1}{2}\left[\frac{(d-1)\delta L}{L}\right]^2\left[\lambda+\frac{2}{d-1}\mu\right]. \tag{9}$$

From this and (8) it follows that

$$\kappa/\mu=\lambda/\mu+2/d=4/d. \tag{10}$$

We now turn to consider a slightly more realistic 2D elastic fractal network model, where the above-mentioned flaw is removed. We use a unit cell that is a three-terminal linear elastic element [see Fig. 2(a)]. A force $\vec{f}$ can be applied at each terminal, and as a result of these forces a displacement $\vec{u}$ appears at each terminal. The relation between the forces and the displacements is chosen so that the unit element represents an isotropic elastic material with arbitrary values of λ and μ:

$$\begin{aligned} f_{1x}^{(1)}&=(\lambda+2\mu)u_{1x}^{(1)}+(\lambda+\tfrac{3}{2}\mu)(u_{2x}^{(2)}+u_{3x}^{(3)})\\ &\quad+\tfrac{1}{2}\sqrt{3}\mu(u_{2y}^{(2)}-u_{3y}^{(3)}),\\ f_{1y}^{(1)}&=-\tfrac{1}{2}\sqrt{3}\mu(u_{2x}^{(2)}-u_{3x}^{(3)})\\ &\quad+\mu(u_{1y}^{(1)}-\tfrac{1}{2}u_{2y}^{(2)}-\tfrac{1}{2}u_{3y}^{(3)}). \end{aligned} \tag{11}$$

The upper indices here refer to the terminal number, while the lower indices define a component axis; there is a different local coordinate system associated with each terminal [see Fig. 2(a)]. Similar equations can be written for $f_{2x}^{(2)}$, $f_{2y}^{(2)}$, $f_{3x}^{(3)}$, and $f_{3y}^{(3)}$.

A fractal network is produced by hooking up these three-terminal elements in the same way as a 2D Sierpinski gasket [see Fig. 2(b), which is the analog of Fig. 1(a)]. At a point of contact between two unit elements [the black circles in Fig. 2(b)], the displacement is common to the two elements, and the forces exerted must be equal and opposite in equilibrium. As in the case of the simple gasket, the network at each stage is connected to the rest of the system only through three external vertices. Therefore an $L=2$ triplet of elements is again exactly equivalent to a single element with renormalized values of λ and μ. A tedious but straightforward calculation leads to the following relations between the $l=1$ and the $l=0$ moduli:

$$\kappa_1=\frac{3\mu_0\kappa_0}{\kappa_0+4\mu_0},\quad \mu_1=\frac{3\mu_0\kappa_0}{5\kappa_0+2\mu_0}, \tag{12}$$

from which we find

$$\kappa_1/\mu_1=(2+5\kappa_0/\mu_0)/(4+\kappa_0/\mu_0). \tag{13}$$

If this transformation is iterated starting from any positive value for κ_0/μ_0, it is easy to see that the result converges upon a fixed point at $\kappa/\mu=2$, in perfect agreement with the expression $4/d$ when $d=2$. The significance of this result is that the ratio κ_0/μ_0 is an irrelevant parameter of the problem for critical properties, and that we could have assumed $\kappa_0/\mu_0=2$ from the outset. In that case, the transformation (12) reduces to $\kappa_1=\kappa_0/2$, $\mu_1=\mu_0/2$, i.e., the same as for the 2D Sierpinski gasket of springs with a *single* spring constant. Thus, the model we discussed earlier is as good as the present one as far as critical behavior is concerned, and leads to the same index T. We suspect that a similar simplification occurs in the d-dimensional generalization of the present model, but this needs to be checked.

At this point it is useful to ask what are the pre-

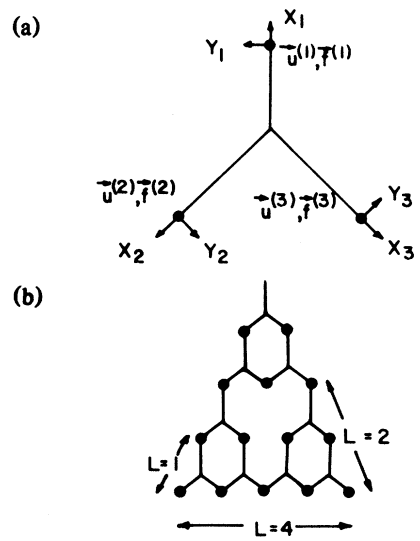

FIG. 2. (a) Three-terminal linear device representing an isotropic elastic material with arbitrary λ and μ. At each terminal there is a displacement $\vec{u}^{(i)}$ and a force $\vec{f}^{(i)}$, $i=1,2,3$, and a local coordinate system (X_i,Y_i). (b) Sierpinski-gasket-like $L=4$ hookup of the unit elements of (a).

dictions of the self-consistent or effective-medium approximation of Hill[5] and Budiansky,[6] suitably generalized to an arbitrary dimensionality d. For a two-component mixture where one component is voids ($\kappa=\mu=0$) we obtain

$$\frac{\kappa_e}{\kappa_0}=\frac{p-\alpha}{1-\alpha},\quad \frac{\mu_e}{\mu_0}=\frac{p-\beta}{1-\beta},$$
$$\alpha=d-\frac{(d-1)(d+2)}{2},\qquad (14)$$
$$\beta=\left[1+\frac{2(d-1)}{d}\frac{\mu_e}{\kappa_e}\right]^{-1}.$$

It is a straightforward matter to show that κ_e and μ_e both have a threshold at $p_c=2/(d+1)$, and that the critical exponent is $T=1$ for both moduli. The same calculation also reveals that $\kappa_e/\mu_e=4/d$ asymptotically for $p\rightarrow p_c$. It is also easy to check that if in (14) we use values for κ_0 and μ_0 that obey $\kappa_0/\mu_0=4/d$, then the results for κ_e and μ_e also obey $\kappa_e/\mu_e=4/d$ *independent of* p. This latter property is also present in the Clausius-Mossotti–type approximation (CMA).[7] The significance of this is that while neither the effective-medium approximation nor the CMA can be relied upon to give correct critical behavior (in particular, the threshold in the CMA is at $p=0$), they are both asymptotically correct when the volume fraction of the voids $1-p\rightarrow 0$. More precisely, they are both exact (and give identical results) to order $1-p$. Thus, the fact that these approximations yield $\kappa_e/\mu_e=4/d$ for $\kappa_0/\mu_0=4/d$, independent of p, proves that this value is a fixed point at least to order $1-p$. This independence might possibly be maintained in higher orders too—all the way down to p_c.

In summary, we have shown strong indications that the critical behavior of the elastic moduli near a percolation threshold differs from that of the conductivity, and that the elasticity critical index T is probably appreciably greater than the conductivity index t. We have also shown indications to support the conjecture that κ_e and μ_e have the universal ratio $4/d$ (especially that $\kappa_e/\mu_e=2$ in 2D), independent of p or of κ_0 and μ_0 near the threshold. Calculations are in progress on simulations of a discrete random elastic network in order to investigate further these interesting results. Experimental investigations of this problem can be conducted on "weak" solids, e.g., weakly compacted sandstone, or polymer solutions near a sol-gel transition. The difference expected between t and T seems to be large enough to be observable even at moderate precision.

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Note added.— After this work was completed, Feng and Sen published results of simulations of an elastic percolating network.[8] They also find critical behavior that is different from that of the electrical conductivity.

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[1]B. B. Mandelbrot, *Fractals: Form, Chance and Dimension* (Freeman, San Francisco, 1977), and *The Fractal Geometry of Nature* (Freeman, San Francisco, 1983).

[2]Y. Gefen, A. Aharony, B. B. Mandelbrot, and S. Kirkpatrick, Phys. Rev. Lett. **47**, 1771–1774 (1981).

[3]P. G. de Gennes, J. Phys. (Paris) Lett. **37**, L1–L2 (1976).

[4]The first of the two terms on the left-hand side of (9) comes from the compression of all the $d(d-1)/2$ bonds belonging to the $(d-1)$-dimensional simplex by the amount δL. The second term comes from the compression of the d remaining bonds of the d-dimensional simplex by the amount $\delta L\,(d-1)/2d$. This is directly related to the ratio $[(d+1)/2d]^{1/2}$ between the height and the edge of a d-dimensional simplex.

[5]R. Hill, J. Mech. Phys. Solids **13**, 213–222 (1965).

[6]B. Budiansky, J. Mech. Phys. Solids **13**, 223–227 (1965).

[7]Y. Kantor and D. J. Bergman, J. Mech. Phys. Solids **30**, 355–376 (1982).

[8]S. Feng and P. N. Sen, Phys. Rev. Lett. **52**, 216–219 (1984).