Is Isotropic Turbulent Diffusion Symmetry Restoring?

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Analytic expressions of the convection diffusivity tensor in fully developed isotropic turbulent flow are given and used to evaluate the broadening of a cloud of marked particle pairs perpendicular and longitudinal with respect to an initial separation. Both turn out to remain different because of the difference between $\langle v_\perp^2 \rangle$ and $\langle v_\parallel^2 \rangle$. Parameter-free expressions for the variances $\sigma_\perp$ and $\sigma_\parallel$ yield the dependence on the Reynolds number and other flow parameters in addition to the known scaling behavior $\propto \tau^{3-\gamma}$.

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The pair separation of marked particles by convection in a turbulent fluid leads to a broadening of an initially sharp distribution. This is called turbulent relative diffusion. A most remarkable feature is the strong dependence of the turbulent diffusivity on the extension of the cloud, first studied by Richardson.[1] Correspondingly, the variance of the particle distribution, denoted by σ, and the diffusivity, defined by $d\sigma/d\tau$, increase strongly with time τ and extension r, respectively. The power-law exponents both of $\sigma \propto \tau^{3+\gamma}$ and of $d\sigma/d\tau \propto r^{4/3+\alpha}$ reflect the fractal nature of turbulence,[2] γ and $\alpha = 2\gamma/(9+3\gamma)$ being the intermittency contributions, and the exponents 3 and $\frac{4}{3}$ following from Kolmogorov–von Weizsäcker–Heisenberg–Onsager scaling. As stressed by several authors (e.g., Mandelbrot and co-workers[2]) turbulent relative diffusion is particularly suited for studying the dynamics of turbulent eddies, especially the probably large effects of intermittency on diffusion.

Recently a unified theory of turbulent relative diffusion was developed[3] which not only deals with the scaling behavior but also yields the magnitude of the variance tensor $\sigma_{ij}(\vec{r},\tau)$. This is defined by

$$\sigma_{ij}(\vec{r},\tau) = \langle \delta R_i(\vec{r},\tau)\delta R_j(\vec{r},\tau)\rangle,$$

where $\delta\vec{R}(\vec{r},\tau) = \vec{R}_\tau(\vec{r}) - \vec{r}$ and $\vec{R}_\tau(\vec{r})$ describes the distances between two particles at time τ, which where released at $\tau = 0$ a distance $r = |\vec{r}|$ apart. The brackets indicate the averaging over all particle pairs. The basic underlying idea is that diffusivity is a transport coefficient which should be intimately connected with the static structure functions of the turbulent flow field, defined by

$$D_{ij}(\vec{r}) = \langle v_i(\vec{r}) v_j(\vec{r})\rangle$$
$$= D_\parallel(r) P_{ij}^\parallel(\vec{r}^0) + D_\perp(r) P_{ij}^\perp(\vec{r}^0),$$

the correlation tensor of the velocity difference $\vec{v}(\vec{r}) = \vec{u}(\vec{x}^0 + \vec{r}) - \vec{u}(\vec{x}^0)$ which we call an $\vec{r}$ eddy. $P_{ij}^{\perp,\parallel}(\vec{r}^0)$ are the projection operators with respect to the unit vector $\vec{r}^0$. The transverse part $D_\perp(r)$ is different from the longitudinal part $D_\parallel(r)$ even if the flow is isotropic (and homogeneous); both are connected by incompressibility, $D_\perp = D_\parallel + rD_\parallel'/2$ $(> D_\parallel)$.

This paper aims at contributing to the question of whether the preferential transverse convective pair separation leads to a nonspherical diffusive broadening of an initial cloud of particle pairs, all released at the same vector distance $\vec{r}$ at $\tau = 0$. In analogy to the static structure function the variance tensor can be decomposed into a longitudinal and a transverse part, $\sigma_\parallel$ and $\sigma_\perp$, according to the relation

$$\sigma_{ij}(\vec{r},\tau) = \sigma_\parallel(r,\tau) P_{ij}^\parallel(\vec{r}^0) + \sigma_\perp(r,\tau) P_{ij}^\perp(\vec{r}^0).$$

As our main result we will show that an isotropic turbulent flow field seems to restore spherical symmetry only approximately; $\sigma_\perp - \sigma_\parallel$ tends to a finite nonzero limit. Its value depends on the form of the radial distribution of the cloud, and is r^2 if the distribution is Gaussian.

Our results are based on a refinement of the closure assumption suggested in Ref. 3. To test its validity we first compare with experimental diffusion data.[4] We then give approximate but analytical expressions for the trace of the variance $\sigma = \sigma_\parallel + 2\sigma_\perp$ and the asymmetry $\sigma_{\rm diff} = \sigma_\perp - \sigma_\parallel$ [cf. Eq. (7)]. Equation (9) presents an asymptotic relation valid for large time τ. We restrict ourselves to the inertial subrange, although extension to initial separations r in the viscous subrange is easily achieved [cf. Ref. 3]. We furthermore treat intermittency using the log-normal model which for the low-order moments needed in turbulent diffusion seems most appropriate. But our formulas also hold for the fractal model if one simply replaces the intermittency exponent μ by 3μ, since as will be shown the intermittency corrections enter only through the second-order structure function.

Following the notation of Ref. 3 the relative turbulent diffusivity is given by

$$d\sigma_{ij}(\vec{r},\tau)/d\tau = d\langle \delta R_i(\vec{r},\tau)\delta R_j(\vec{r},\tau)\rangle/d\tau = \int_0^\tau d\tau' [\langle v_i(\vec{r},\tau)v_j(\vec{r},\tau-\tau')\rangle + (i \rightleftarrows j)]. \quad (1)$$

Both particles perform Lagrangian motion because the Kubo formula (1) contains the Lagrangian velocity difference $\vec{v}(\vec{r},\tau) = d\vec{R}_\tau/d\tau$. The phenomenological closure assumptions are (i) to replace the Lagrangian velocity $\vec{v}(r,\tau)$ of the pair with initial distance $\vec{r}$ after time τ by $\vec{v}(r_\tau, 0)$, the velocity difference at the rms separation r_τ; (ii) to decompose the time correlation function $\langle v_i(r_\tau,0) \times v_j(r_\tau,\tau')\rangle$ into its longitudinal and transverse part with respect to the direction of $\vec{R}_\tau$, given by the unit vector $\vec{R}_\tau^0$, and to average the directional projectors, e.g.,

$$\langle R^0_{\tau,i}(\vec{r})R^0_{\tau,j}(\vec{r})\rangle_{\parallel} = \langle(\vec{R}^0_\tau \cdot \vec{r}^0)^2\rangle = \langle\cos^2\alpha\rangle;$$

(iii) to use the relaxation approximation[5] for the temporal decay of the correlation function

$$\langle v_i(r_\tau,0)v_j(r_\tau,\tau')\rangle_{\parallel,\perp} = D_{\parallel,\perp}(r_\tau)e^{-\tau'/t_{\parallel,\perp}(r_\tau)}, \quad (2)$$

where $t_{\parallel,\perp}(r_\tau)$ is the correlation time of an eddy with extension r_τ and has been evaluated in Ref. 5 exactly, based on the Navier-Stokes equation,

$$t_{\parallel,\perp} = 3D_{\parallel,\perp}/2\epsilon, \quad (3)$$

for r in the inertial subrange with ϵ denoting the mean rate of energy dissipation; and (iv) to express the rms separation of the particles at τ by the variance, $r_\tau = [r^2 + \sigma(r,\tau)]^{1/2}$. This yields from Eq. (1) a coupled set of first-order differential equations for the variances $\sigma_\parallel, \sigma_\perp$, which we solved numerically to compare with the data[4] (see Fig. 1). For the static structure function

$$D_\parallel(r) = b_\parallel N_{\mathrm{Re}\lambda}^{-\mu/6}(r/\eta)^{2/3+\mu/9}(\epsilon\nu)^{1/2} \quad (4)$$

was used. $N_{\mathrm{Re}\lambda}$ is the Taylor Reynolds number, μ the Kolmogorov intermittency exponent, $\mu = 0.25$ from the BOMEX data,[6] $b_\parallel = 3$, $\eta = (\nu^3/\epsilon)^{1/4}$ (here 0.19 cm) the Kolmogorov length, and ν the kinematic viscosity; $\nu = 0.153$ cm^2/s for air.

We now write down the formulas underlying the theoretical curves of Fig. 1, obtained from (1) by use of (i) through (iv). It is convenient to scale them properly, lengths by r, times by $t_\parallel(r)$ (the decay time of eddies of the initial size r): $\Gamma_{\parallel,\perp} = \sigma_{\parallel,\perp}/r^2$ dimensionless variance; $t = \tau/t_\parallel(r)$ dimensionless time.

The τ' integral in (1) yields a factor (cf. Ref. 3) $[1 - \exp - t/(1+\Gamma)^{1/3+\mu/18}]$, being $\propto t$ for small times. But the trace $\Gamma = \Gamma_\parallel + 2\Gamma_\perp$ turns out to increase so fast, $\Gamma \cong (t)^{3+\gamma}$, that the exponent is $\cong a^{-1} << 1$ for all times. Thus in contrast to thermal diffusion the correlation never really decays and the diffusivity is proportional to t and not proportional to relaxation time, unless a can be made small. Also, in contrast to molecular diffusion the magnitude of the diffusivity is not $\langle \vec{v}^2\rangle = 3kT/$

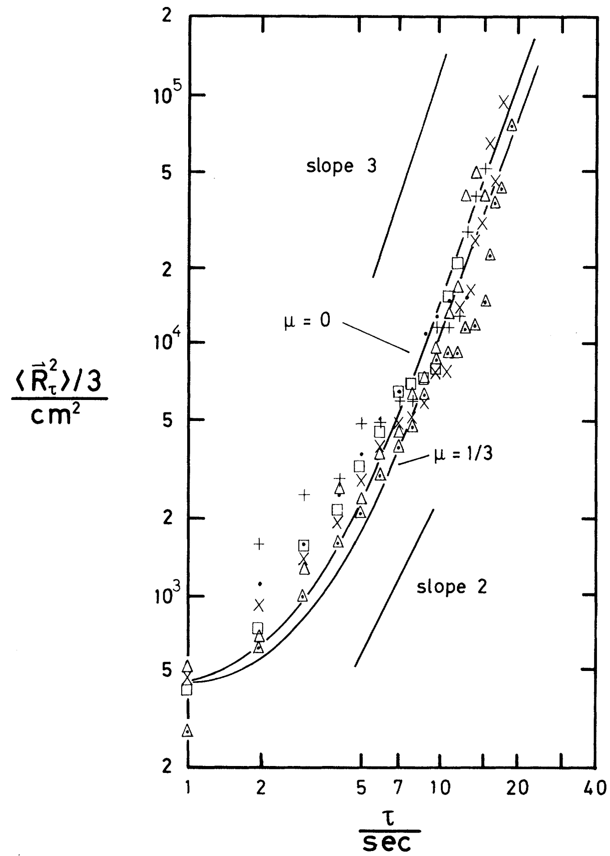

FIG. 1. Increase of the mean square extension of the cloud in one direction $\langle\vec{R}_\tau^2\rangle/3 = r^2 + \sigma(\tau)/3$ vs time τ (full curves). The experimental data are the measurements of Frenkiel and Katz [releasing a smoke puff of gun powder to realize the (spherical) initial cloud with $r/\eta \cong 100$] with Gifford's analysis. The paramaters we took from Ref. 4 are mean velocity $U = 600$ cm/s, height $z = 20$ m, and viscosity $\nu = 0.153$ cm^2/s; hence $N_{\mathrm{Re}\lambda} \cong 0.9(zU/\nu)^{1/2} = 2600$. Estimated energy dissipation $\epsilon \cong 2.2$ cm^2/s^3; hence turbulence strength $u' = (\epsilon\nu N^2_{\mathrm{Re}\lambda}/15)^{1/4} = 20$ cm/s and microscale $\lambda = \nu N_{\mathrm{Re}\lambda}/u' = 20$ cm.

$m=$const, but *increases* with the particle distance r_τ since more energy is in the larger eddies. This is the basic physics behind the anomalously enhanced convective diffusion. It is

$$a=[\tfrac{11}{2}(1-\mu/12)(1+\mu/33)b_{\parallel}^{3}]^{1/2}\,[N_{\mathrm{Re}\lambda}^{-3/2}(r/\eta)]^{\mu/6}, \tag{5}$$

so that $a^{-1}(\mu=0)=0.08$ and $a^{-1}(\mu=0.5)=0.15$. As a function of time the exponent decreases like $t^{-\gamma}$; $\gamma=3\mu/(12-\mu)$. By expanding the exponential the resulting equation for the trace Γ can be solved analytically:

$$\Gamma(t)=(1+a^2t^2)^{3/2(1-\mu/12)}-1. \tag{6}$$

We think that this is the first explicit and parameter-free formula which expresses the increase of the variance for all times in its dependence on the initial size r, the Reynolds number $N_{\mathrm{Re}\lambda}$, intermittency exponent μ, energy dissipation ϵ, and kinematic viscosity ν (via $t_{\parallel}$). There is a t^2 regime for small times and a $t^{3+\gamma}$ regime for large times. The comparison with the data is promising.

If molecular viscosity is added (cf. Ref. 3), instead of the closed expression (6) the differential equations have to be integrated numerically. Extension to two-dimensional turbulence is also easy and straightforward.

While the equation for the trace σ or Γ does not depend on the directional factor at all, the latter determines the equation for the asymmetry $\Gamma_{\mathrm{diff}}=\Gamma_{\perp}-\Gamma_{\parallel}$ which can be written in the form

$$\frac{d\Gamma_{\mathrm{diff}}}{d\Gamma}=\frac{1}{11}\,\frac{1+\mu/6}{1+\mu/33}\,\frac{3\langle\cos^2\alpha\rangle-1}{2}. \tag{7}$$

This determines Γ_{diff} in terms of Γ, since $\langle\cos^2\alpha\rangle$ is expected to depend on the form of the cloud, i.e., on $\Gamma_{\parallel}$ and $\Gamma_{\perp}$. Certainly it is $\langle\cos^2\alpha\rangle=1$ initially, so that Γ_{diff} increases much slower, $\cong\frac{1}{11}$, than Γ. This small factor is precisely $(D_{\perp}-D_{\parallel})/D=\kappa/(2\kappa+6)$ with $\kappa=d(\ln D_{\parallel})/d(\ln r)$. Thus, the r exponent κ of the static structure function can be measured through the initial growth of the asymmetry, which therefore is universal. In the long run $\langle\cos^2\alpha\rangle$ tends to $\frac{1}{3}$. Depending on the manner in which the right-hand side of (7) tends to zero, Γ_{diff} approaches a *finite* value or increases logarithmically.

If the probability distribution $P(\vec{R})$ for the pair distances $\vec{R}_\tau$ is a function of the variable $x^2=(R_x^2+R_y^2)/\sigma_{\perp}+(R_z-r)^2/\sigma_{\parallel}$ only, it is possible to expand the directional average systematically in powers of $1/\Gamma$, the dimensionless inverse trace r^2/σ. We found in lowest order

$$\langle\cos^2\alpha\rangle=\tfrac{1}{3}+\tfrac{2}{5}(\Delta-\Gamma_{\mathrm{diff}})/\Gamma. \tag{8}$$

$\Delta=1$ for a Gaussian pair distribution and, in general, $\Delta=\langle x^{-2}\rangle_P$. Using (8) we get from (7)

$$\Gamma_{\perp}-\Gamma_{\parallel}=\Delta-k_1/\Gamma^{\delta_1}=\Delta-k_2/t^{\delta_2}. \tag{9}$$

The constants k_1 and k_2 are determined by $\Gamma_{\mathrm{diff}}\,(\Gamma)$ at a reference point. The exponents are rather small,

$$\delta_1=\tfrac{3}{55}(1+\mu/6)/(1+\mu/33),\qquad \delta_2=3\delta_1/(1-\mu/12)\cong3\delta_1. \tag{10}$$

Hence a *finite* deviation Δ from a spherical form remains which is approached rather slowly. The direction perpendicular to $\vec{r}$ is pronounced imaging the preference of perpendicular flow due to incompressibility. The *relative* deviation $\Gamma_{\mathrm{diff}}/\Gamma$, of course, approaches zero $\propto t^{3+\gamma}$, describing the symmetry-restoring effect of the isotropic turbulent flow field. At present, the finite asymptotic asymmetry Δ is beyond the scope of existing data; better measurements would provide useful information about the marked particle pairs' distribution function, especially its inverse-squared moment.

Finally, we emphasize that we considered r and r_τ in the inertial subrange. If r_τ increases beyond the outer turbulent scale $L=\eta N_{\mathrm{Re}\lambda}^{3/2}$, we expect $D_{\parallel}$ to become constant, $2u'^2$; hence $t_{\parallel}$ stays constant, the correlation in (1) decays, and the diffusion becomes normal, $\sigma(\tau)\cong6D_{\parallel}(L)t_{\parallel}(L)\tau=2K\tau$. For the convective diffusion constant in ranges beyond L we evaluated $K=\frac{6}{5}\nu N_{\mathrm{Re}\lambda}^2$, which is of the order of 300 $\mathrm{m^2\,s^{-1}}$ for atmospheric turbulence.

We hope this discussion might stimulate more measurements and analysis of data for convective turbulent diffusion in terms of the formulas (6) and (9). If the close connection between the rather well-established structure functions $D_{\perp,\parallel}$ and the turbulent diffusion provided by our closure is confirmed, the simple formulas for the transport of particle waste in air or water may find useful applications. They may also serve to clarify the appropriate model for intermittency by diffusivity measurements as was emphasized already by Procaccia and co-workers.[2] The absolute curdling model gives even stronger effects on diffusion than the log-normal model which we used here, as substantiated earlier.

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