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Quantum Stress Tensor in Schwarzschild Space-Time

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The vacuum expectation value of the stress-energy tensor for the Hartle-Hawking state in Schwarzschild space-time has been calculated for the conformal scalar field. $\langle T_\mu^\nu \rangle$ separates naturally into the sum of two terms. The first coincides with an approximate expression suggested by Page. The second term is a "remainder" that may be evaluated numerically. The total expression is in good qualitative agreement with Page's approximation. These results are at variance with earlier results given by Fawcett whose error is explained.

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The computation of the vacuum expectation value of the stress-energy tensor for Schwarzschild space-time has presented a challenge since the celebrated calculation by Hawking of black-hole radiance. The original motivation, to understand where the particles came from, has to some extent been superceded by a clear understanding of the interplay between "real particles" and "vacuum polarization effects."[1] The picture that has emerged is that, although the density of stress energy associated with the emerging particles is infinite at the horizon, so is the density of stress energy associated with the vacuum polarization. The two effects contribute with opposite sign and the net value of stress-energy density remains finite as the horizon is approached. Although the finiteness of the expectation value of the stress tensor at the horizon is no longer an issue, there remains the problem of understanding its detailed structure in terms of the geometry of the Schwarzschild manifold. As a necessary step in this direction we present here the results of a calculation of $\langle T_\mu^\nu \rangle$ for a conformal scalar field in the Hartle-Hawking vacuum for the region exterior to the horizon.

Suggestions that some sort of semiclassical description of $\langle T_\mu^\nu \rangle$ in terms of the background geometry may be possible are provided by estimates[2,3] of $\langle T_\mu^\nu \rangle$ made on the basis of the properties of the Schwarzschild metric under conformal transformations. An intuitive understanding of the structure of $\langle T_\mu^\nu \rangle$ in terms of the geometry of the background would be of considerable interest with regard to the structure of solutions of the semiclassical Einstein equation

$$R_\mu^\nu - \tfrac{1}{2} R g_\mu^\nu = -8\pi \langle T_\mu^\nu \rangle,$$

and with regard to vacuum energy in Kaluza-Klein theories, the subject of much current work.

The operator expression for the expectation value of the stress-energy operator,

$$\langle T_\mu^\nu \rangle = \langle (\tfrac{2}{3}\phi_{;\mu}\phi_{;}{}^{\nu} - \tfrac{1}{6} g_\mu^\nu \phi_{;\alpha}\phi_{;}{}^{\alpha} - \tfrac{1}{3}\phi\phi_{;\mu}{}^{\nu}) \rangle,$$

was renormalized by means of the covariant point separation procedure of DeWitt[4] and Christensen.[5] In terms of the propagator appropriate to this state

$$G(x,x') = i \langle \phi(x)\phi(x') \rangle$$

that satisfies

$$\Box G(x,x') = -g^{-1/2}\delta(x,x'),$$

the renormalized value of $\langle T_\mu^\nu \rangle$ is given by the expression

$$\langle T_\mu^\nu \rangle_{\mathrm{REN}} = \lim_{x' \to x} \{ -i [\tfrac{1}{3} (G_{;\mu\alpha'} g^{\alpha'\nu} + G_{;}{}^{\nu}{}_{\alpha'} g^{\alpha'}{}_{\mu}) - \tfrac{1}{6} G_{;\alpha\beta'} g^{\alpha\beta'} g_\mu^\nu - \tfrac{1}{6} (G_{;\mu}{}^{\nu} + G_{;\alpha'\beta'} g^{\alpha'}{}_{\mu} g^{\beta'\nu})] - \langle T_\mu^\nu \rangle_{\mathrm{subtract}} \},$$

where $\langle T_\mu^\nu \rangle_{\mathrm{subtract}}$ are the subtraction terms of Christensen.[5]

In principle, the above equation is easy to evaluate. One must only solve the wave equation for the propagator as a sum over mode functions, perform the indicated subtraction, and sum the resultant convergent expression for $\langle T_\mu^\nu \rangle_{\mathrm{REN}}$. In practice, a number of difficulties present themselves during this process. For example, some subtlety must be employed in performing the subtraction. The subtraction term must be rewritten as a sum which is compatible with the mode-sum expression for the propagator so that a convergent sum will be obtained when the limit is taken. The summation of the resultant expression for $\langle T_\mu^\nu \rangle_{\mathrm{REN}}$ is further complicated by the fact that, as the radial solutions are not expressible in terms of well-known functions, a combination of approximation and numerical analysis must be applied. Our calculation, the details of which will be presented elsewhere, reveals that in Schwarzschild coordinates $\langle T_\mu^\nu \rangle_{\mathrm{REN}}$ has the form

$$\langle T_\mu^\nu \rangle_{\mathrm{REN}} = \frac{\pi^2}{90(8\pi M)^4} \left\{ \frac{1-(2M/r)^6(4-6M/r)^2}{(1-2M/r)^2} \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + 24 \left(\frac{2M}{r} \right)^6 \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + 1920 \Delta_\mu^\nu \right\}.$$

The first two terms correspond precisely to the approximation which Page[2] obtained by means of a Gaussian approximation to the proper-time propagator.[6] Being an approximation of WKB type Page's expression might reasonably be expected to furnish a good approximation to the true quantity, particularly in view of the fact that the corresponding approximation[7,8] to $\langle \phi^2 \rangle$ agrees with the true quantity to better than 1%.

The components of the remainder term Δ_μ^ν involve sums over solutions to the radial equation

$$\left\{ \frac{d}{d\xi} (\xi^2 - 1) \frac{d}{d\xi} - l(l+1) - \frac{n^2 (1+\xi)^4}{16(\xi^2-1)} \right\} R(\xi) = 0,$$

TABLE I. The sums that appear in Δ_μ^ν.

$$S_1 = \frac{1}{16} \left(\frac{\xi+1}{\xi-1} \right) \sum_{n=1}^{\infty} n^2 \left\{ \sum_{\ell=0}^{\infty} \left[\frac{2\ell+1}{n} p_\ell^n(\xi) q_\ell^n(\xi) - \frac{2}{(\xi^2-1)^{1/2}} \right] + \frac{n}{2} \left(\frac{\xi+1}{\xi-1} \right) \right\}$$

$$S_2 = \left(\frac{\xi-1}{\xi+1} \right) \sum_{n=1}^{\infty} \left\{ \sum_{\ell=0}^{\infty} \left[\frac{2\ell+1}{n} \frac{dp_\ell^n(\xi)}{d\xi} \frac{dq_\ell^n(\xi)}{d\xi} + \frac{2\ell(\ell+1)}{(\xi^2-1)^{3/2}} + \frac{n^2(\xi+1)^4}{16(\xi^2-1)^{5/2}} - \frac{3}{4(\xi^2-1)^{5/2}} \right] - \frac{n^3(\xi+1)^3}{96(\xi-1)^3} + \frac{n}{3(\xi-1)^3} \right\}$$

$$S_3 = \frac{1}{(\xi+1)^2} \sum_{n=1}^{\infty} \left\{ \sum_{\ell=0}^{\infty} \left[\frac{2\ell+1}{n} (\ell+\tfrac{1}{2})^2 p_\ell^n(\xi) q_\ell^n(\xi) - \frac{2(\ell+\frac{1}{2})^2}{(\xi^2-1)^{1/2}} + \frac{n^2(1+\xi)^4}{16(\xi^2-1)^{3/2}} - \frac{1}{4(\xi^2-1)^{3/2}} \right] - \frac{n^3(\xi+1)^4}{48(\xi-1)^2} + \frac{n}{24(\xi-1)^2} (3\xi^2 - 8\xi + 13) \right\}$$

$$S_4 = \frac{1}{(\xi+1)^2} \frac{\partial}{\partial\xi} \sum_{n=1}^{\infty} \left\{ \sum_{\ell=0}^{\infty} \left[\frac{2\ell+1}{n} p_\ell^n(\xi) q_\ell^n(\xi) - \frac{2}{(\xi^2-1)^{1/2}} \right] + \frac{n(\xi+1)}{2(\xi-1)} \right\}$$

$$S_5 = \frac{1}{4(\xi+1)^2} \sum_{n=1}^{\infty} \left\{ \sum_{\ell=0}^{\infty} \left[\frac{2\ell+1}{n} p_\ell^n(\xi) q_\ell^n(\xi) - \frac{2}{(\xi^2-1)^{1/2}} \right] + \frac{n}{2} \left(\frac{\xi+1}{\xi-1} \right) \right\}$$

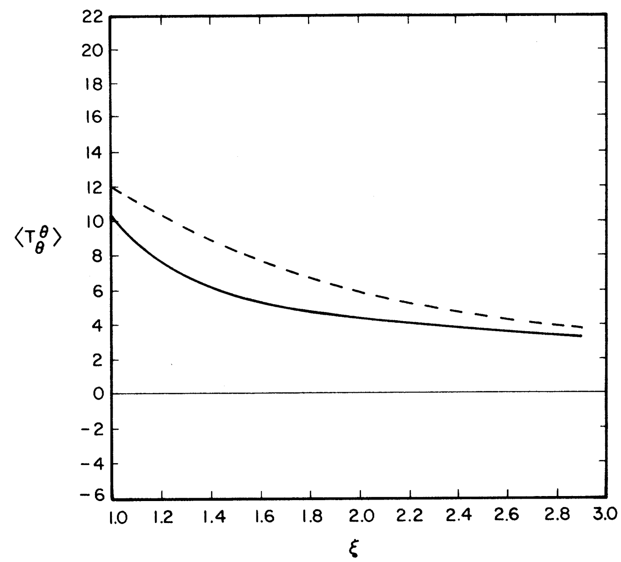

FIG. 1. $[90(8\pi M)^4/\pi^2]\langle T^\theta_\theta\rangle$ as a function of $\xi = r/M - 1$. The dashed line represents Page's approximation.

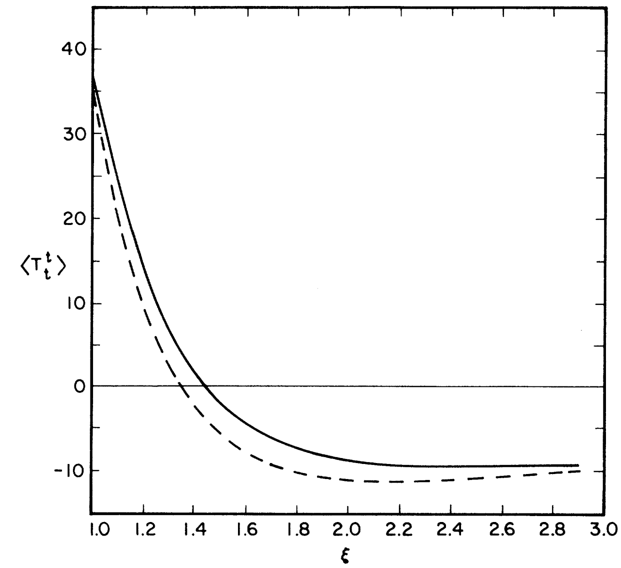

FIG. 2. $[90(8\pi M)^4/\pi^2]\langle T^t_t\rangle$ as a function of $\xi = r/M - 1$. The dashed line represents Page's approximation.

which is expressed in terms of a transformed radial coordinate

$$\xi = r/M - 1.$$

They involve the functions $p_l^n(\xi)$ and $q_l^n(\xi)$ that satisfy the radial equation subject to the boundary conditions, for $n > 0$, that

$$p_l^n(\xi) \sim (\xi - 1)^{n/2} \text{ as } \xi \to 1,$$

while

$$q_l^n(\xi) \sim (\xi - 1)^{-n/2} \text{ as } \xi \to 1$$

and tends to zero as $\xi \to \infty$.

Specifically, we find

$$\Delta_t^t = 5S_1 - S_2 - S_3 - S_4 + S_5,$$

$$\Delta_r^r = -3S_1 + 3S_2 - 3S_3 + (2\xi - 1)S_4 + 3S_5,$$

$$\Delta_\theta^\theta = \Delta_\phi^\phi = -S_1 - S_2 + 2S_3 - (\xi - 1)S_4 - 2S_5,$$

where the sums S_l are exhibited in Table I.

Numerical evaluation of the $S_l(\xi)$ yields the values depicted in Figs. 1–3. We employ the $(-,+,+,+)$ signature and units in which $\hbar = c = G = k = 1$. With this signature $\langle T_t^t\rangle$ has sign opposite to that of the energy density. As $r \to \infty$ all the curves approach the values appropriate to a thermal bath at the Hawking temperature $(8\pi M)^{-1}$.

It is evident from the figures that Δ_μ^ν does not significantly alter the character of the curves expected on the basis of Page's approximation. These results, however, are in definite disagreement with numerical values previously given by Fawcett,[9] which purported to show that the true value of $\langle T_\mu^\nu\rangle$ differed in important respects from Page's approximation.

Fawcett's error occurs prior to performing numerical analysis. At an early stage of calculation

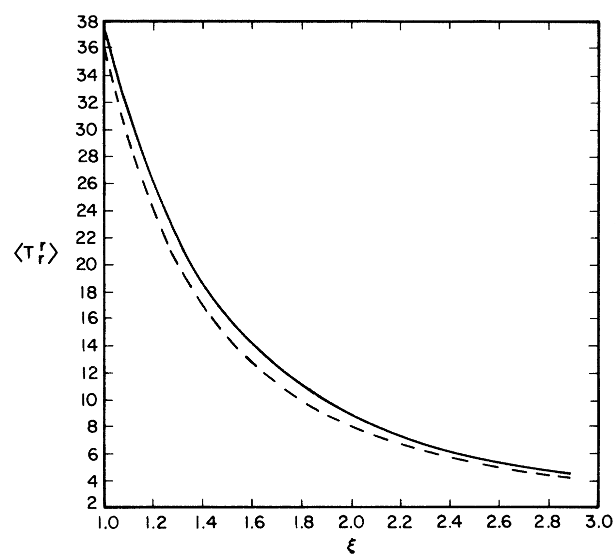

FIG. 3. $[90(8\pi M)^4/\pi^2]\langle T^r_r\rangle$ as a function of $\xi = r/M - 1$. The dashed line represents Page's approximation.

Fawcett[10] effectively writes

$$\langle T_{\mu\nu}\rangle_{\rm REN} = \langle (\phi_{;\mu}\phi_{;\nu} + \tfrac{1}{4} g_{\mu\nu}\phi\Box\phi)\rangle + (6\pi^2)^{-1} g_{\mu\nu} a_2 - \tfrac{1}{12} g_{\mu\nu}\Box\langle\phi^2\rangle - \tfrac{1}{6}\langle\phi^2\rangle_{;\mu\nu},$$

which he employs to evaluate $\langle T_{\theta\theta}\rangle_{\rm REN}$ from which the other components of $\langle T_{\mu\nu}\rangle_{\rm REN}$ are obtained. In his calculation, he assumes that the last term

$$-\tfrac{1}{6}\langle\phi^2\rangle_{;\theta\theta}$$

vanishes on the grounds that $\langle\phi^2\rangle$ is a function of r only. However,

$$\langle\phi^2\rangle_{;\mu\nu} = \langle\phi^2\rangle_{,\mu\nu} - \Gamma_{\mu\nu}{}^{\lambda}\langle\phi^2\rangle_{,\lambda},$$

and therefore

$$\langle\phi^2\rangle_{;\theta\theta} = -\Gamma_{\theta\theta}{}^{r}\frac{\partial}{\partial r}\langle\phi^2\rangle,$$

which is nonzero. If this term is included, then Fawcett's results agree with ours to within numerical error.

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[1]For a review of the physical interpretation of the Hartle-Hawking state and of the other vacua appropriate to Schwarzschild space-time, the reader is referred to D. W. Sciama, P. Candelas, and D. Deutsch, Adv. Phys. **30**, 327 (1981), which contains references to the original literature.

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