Topological Quantum Effects for Neutral Particles

Y. Aharonov[(a)] and A. Casher
Department of Physics and Astronomy, Tel Aviv University, Tel Aviv 69978, Israel
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We derive the effective Lagrangian which describes the interaction between a charged particle and a magnetic moment in the nonrelativistic limit. It is shown that neutral particles with a magnetic moment will exhibit the Aharonov-Bohm effect in certain circumstances. We suggest several types of experiments.

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Topological effects in quantum mechanical systems are manifested through the generation of relative phases which accumulate on the wave function of a particle moving through a non-simply-connected force-free region. The generic phenomenon of this type is the Aharonov-Bohm (A-B) effect[1] which is due to the presence of a vector potential in the Lagrangian of the particle:

$$L=\tfrac{1}{2}Mv^2+e\vec{A}\cdot\vec{v}, \tag{1}$$

which leads to an interference effect due to the phase:

$$\exp iS=\exp i\oint\vec{A}\cdot\vec{v}\,dt=\exp i\oint\vec{A}\cdot d\vec{x}. \tag{2}$$

The Lagrangian (1) is that of a charged particle in an external electromagnetic vector potential. Is it possible to generate a situation in which a neutral particle exhibits the A-B effect? We will show that this is indeed possible and is actually a necessary consequence of the physics described by Eq. (1).

Consider first a solenoid located at the point $\vec{R}$ (in a plane) which interacts with a charged particle at $\vec{r}$. Since the solenoid is electrically neutral it would seem that the Lagrangian of the system is

$$L\overset{?}{=}\tfrac{1}{2}mv^2+\tfrac{1}{2}MV^2+e\vec{A}(\vec{r}-\vec{R})\cdot\vec{v}. \tag{3}$$

Equation (3) is wrong since it leads to a nonzero force on the charged particle

$$m\dot{v}_j=ev_i(\partial_iA_j-\partial_jA_i)-eV_i\partial_iA_j.$$

The first term on the right-hand side is the ordinary magnetic Lorentz force and vanishes outside the solenoid. The second term, however, is proportional to the velocity of the solenoid and is moreover not gauge invariant and nonzero. Thus we propose to correct the Lagrangian (3) by an extra term:

$$L=\tfrac{1}{2}mv^2+\tfrac{1}{2}MV^2+e\vec{A}(\vec{r}-\vec{R})\cdot(\vec{v}-\vec{V}). \tag{4}$$

It is readily checked that Eq. (4) leads to a gauge-invariant force which vanishes with the magnetic field:

$$m\dot{\vec{v}}=e(\vec{v}-\vec{V})\times\vec{B}=-M\dot{\vec{V}};\qquad \vec{B}=\nabla\times\vec{A}(\vec{r}-\vec{R}). \tag{5}$$

Note that Eqs. (4) and (5) are Galilean and translation invariant and lead to conservation of the momentum:

$$\vec{p}+\vec{P}=(\vec{p}-e\vec{A})+(\vec{P}+e\vec{A})=m\vec{v}+M\vec{V}. \tag{6}$$

We shall now prove that Eq. (4) actually follows from electrodynamics and correctly describes the interaction between a charged particle and a neutral source of a magnetic field. A solenoid may be viewed as a line of magnetic moments. Consider a magnetic moment $\vec{\mu}$ moving in an external static electric field. We work in the radiation gauge $\nabla\cdot\vec{A}=0$. A magnetic moment is generated by a current distribution:

$$\vec{j}=\nabla\times\vec{\mathscr{M}};\quad \int\vec{\mathscr{M}}=\vec{\mu}. \tag{7}$$

Lorentz invariance decrees that (to order v/c) a moving magnetic moment generates a charge density[2] (our units are $\hbar=c=1$):

$$\rho=\vec{V}\cdot\vec{j}. \tag{8}$$

The Lagrangian of our magnetic moment is thus

$$L=\tfrac{1}{2}MV^2-\int\rho A_0=\tfrac{1}{2}MV^2-\int A_0\vec{V}\cdot\nabla\times\vec{\mathscr{M}}=\tfrac{1}{2}MV^2-\vec{V}\cdot\vec{E}\times\vec{\mu}, \tag{9}$$

where $\vec{E}=-\nabla A_0$ is the electric field. If, however, $\vec{E}$ is the Coulomb field of a charged point particle then

$$\vec{E}\times\vec{\mu}=\frac{e}{4\pi}\frac{(\vec{R}-\vec{r})\times\vec{\mu}}{|R-r|^3}=e\vec{A}(\vec{r}-\vec{R}), \tag{10}$$

where $\vec{A}$ is the vector potential at the point $\vec{r}$

whose source is the magnetic moment $\vec{\mu}$ at R! Adding the Lagrangian of the charged particle we thus recover Eq. (4). Note that a gauge transformation $\vec{A} \to \vec{A} + \nabla f(r-R)$ changes L by the total time derivative $\dot{f}$. Before we turn to some applications of Eq. (4) we pause to reinterpret the momentum conservation equation (6). In fact, using Eq. (10) and the fact that inside the solenoid $\vec{B} = \vec{\mu}$ we easily find that Eq. (6) is

$$\vec{p} + \vec{P} = m\vec{v} + \vec{P} + \int \vec{E} \times \vec{B}. \qquad (11)$$

Hence, the correction which restores the momentum conservation is the field momentum carried along with the solenoid. We also record here the generalization of Eq. (4) for a system of charged particles interacting with magnetic multipoles:

$$L = \tfrac{1}{2} \sum m_e v_e^2 + \tfrac{1}{2} \sum M_m V_m^2 + \sum_{e,m} (\vec{v}_e - \vec{V}_m) \cdot e\vec{A}(\vec{r}_e - \vec{R}_m). \qquad (12)$$

(Purely electric-electric terms were left out.)

Let us turn now to the application of the Lagrangian (12) to quantum mechanics. Because of the dependence of A on $(\vec{r} - \vec{R})$, the system possesses a kind of duality when the roles of $\vec{r}_e$ and $\vec{R}_m$ are reversed. In particular, a magnetic moment moving in the field of a straight homogeneously charged line feels no force and undergoes an A-B effect; the A-B phase is

$$S_{AB} = -\oint e\vec{A}(\vec{r} - \vec{R}) \cdot d\vec{R} = \mu\lambda \qquad (13)$$

where λ is the charge per unit length on the line and μ the projection of the magnetic moment along the line. It is convenient to express λ and μ in terms of the relevant length and mass scales:

$$\lambda = e/\xi \quad \mu = ge/2m. \qquad (14)$$

We thus have (restoring $\hbar$ and c):

$$S_{\rm AB} = 2\pi\alpha g\hbar/mc\xi. \qquad (15)$$

Here α is the fine-structure constant and $\hbar/mc$ is the Compton wave length associated with m; $g = O(1)$ is the g factor. For a neutron $\hbar/mc = 2\times 10^{-14}$ cm, while for an atom $\hbar/mc$ is controlled by the electron mass and is $\sim 4\times 10^{-11}$ cm. Thus, in order to get observable A-B phases ($S_{\rm AB} \sim \pi/2$) we need linear charge densities

$$\text{neutron: } \lambda \sim e/10^{-15} \text{ cm}, \qquad \text{atom: } \lambda \sim e/2\times 10^{-12} \text{ cm}. \qquad (16)$$

While these densities seem huge we remark that there is no limitation of principle on the thickness of the charged line so that experimental verification with neutrons and/or atomic beams might become feasible. Note also that the magnetic-moment beam should be polarized and the effect is maximal when the polarization is directed along the charged line.

The e-μ duality mentioned above means that a charged line will act on a superfluid made of bosons which have a magnetic moment in the same way magnetic flux acts on an electric superconductor. In particular, if such a line is passed through a superfluid ring of radius R, Eq. (14) predicts that the superfluid will rotate. The velocity is given by

$$v = (\hbar/2MR)S/\pi \quad \text{for } |S/\pi| < 1, \qquad (17)$$

where M is the mass of the boson. Note that v is a periodic function of the linear charge density with a period given by

$$\Delta(e\xi^{-1}) = 2\pi c/\mu. \qquad (18)$$

Again the effect is maximized if the fluid is magnetically polarized along the direction of charged line.

We return now to the Lagrangian (4). In order to strengthen the foundation of Eq. (4) we shall rederive it from the Dirac equation. The Dirac Lagrangian of a purely magnetic neutral particle is

$$\mathcal{L} = \bar{\psi}[i\not{\partial} - m - \tfrac{1}{2}\mu F^{\mu\nu}\sigma_{\mu\nu}]\psi. \qquad (19)$$

Choosing the representation

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \alpha^k = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix}$$

and writing $\psi = \binom{u}{v}$ we easily find in the nonrelativistic limit the effective Hamiltonian for a magnetic moment in an external electric field:

$$H_{\rm NR} = (1/2m)\vec{\sigma} \cdot (\vec{p} - i\mu\vec{E})\vec{\sigma} \cdot (\vec{p} + i\mu\vec{E}). \qquad (20)$$

Expanding the product of Pauli matrices we have $(\vec{\mu} = \mu\vec{\sigma})$:

$$H_{\rm NR} = (1/2m)(\vec{p} - \vec{E} \times \vec{\mu})^2 - \mu^2 E^2/m \qquad (21)$$

The first term is precisely the Hamiltonian which corresponds to Eq. (9). The second is a correction due to the appearance of an induced electric dipole moment and may be neglected so long as $\mu E << mv$.

Finally we shall exhibit another aspect of the effect by deriving it in a special context. Consider a (2+1)-dimensional Higgs system[3] (a relativistic superconductor) described by the Lagrangian

$$L = -\tfrac{1}{4}F_{\mu\nu}^2 + |(\partial_\mu - ieA_\mu)\phi|^2 - y^2(|\phi|^2 - v^2)^2. \quad (22)$$

As is well known, the U(1) gauge symmetry is spontaneously broken and the photon gets a mass $M=\sqrt{2}ev$. This system has a soliton which describes a localized quantized magnetic fluxon[4] whose field strength decreases exponentially outside the core:

$$e\vec{A}_{cl}(\vec{r}-\vec{R}) = \frac{\hat{z}\times(\vec{r}-\vec{R})}{|\vec{r}-\vec{R}|^3} + O(e^{-M|\vec{r}-\vec{R}|}), \quad (23)$$

($\vec{r}$ is a two-dimensional vector and $\hat{z}$ is the unit vector perpendicular to the plane). Suppose we add an external charge density J_0 so that

$$L \to L - \int A_0 J_0. \quad (24)$$

In order to find A_0 we may neglect all the excitations (photons of mass M and scalars of mass yv) and keep only the collective center of mass coordinates $\vec{R}$. The charge-density operator ρ of the system is

$$\rho = ie(\phi - ieA_0\phi)\phi^* + \text{H.c.} \quad (25)$$

Substituting $\phi(\vec{r}) = \phi_{cl}(\vec{r}-\vec{R})$ and using the field equations

$$-\nabla^2 A_0 = \rho \quad (26)$$

$$-\nabla^2 \vec{A}_{cl} = ie(\nabla\phi_{cl} - ie\vec{A}_{cl}\phi_{cl})\phi_{cl}^* + \text{H.c.}, \quad (27)$$

we find

$$L \to L - \dot{\vec{R}}\cdot\int d\vec{r}\,\vec{A}(\vec{r}-\vec{R})J_0(\vec{r}) \quad (28)$$

which is precisely Eq. (4). This is of course not surprising since it is simply a realization of the previous derivation. Note, however, that if the external charge is viewed as a source, the electric field it generated must be screened by the system so that the total electric field seen by the fluxon is $O(\exp(-M|\vec{r}-\vec{R}|))$. It is remarkable that the only piece of the field $\vec{E}$ which actually enters the effective Lagrangian is the unscreened Coulomb field of the external charge. The consistency of this result is explained by remarking that the term $\vec{\nabla}\cdot\vec{E}\times\vec{\mu}$ generates no force and is only effective in inducing an A-B phase. The particles of the medium are, however, quantized in the proper unit pertaining to the flux so that their share of the phase is $2\pi n$ and hence irrelevant. This phenomena may be summarized by the statement that the superconductor screens all the moments of $\vec{E}$ but does not screen the topological effect of $\exp\oint i\,d\vec{r}\times\vec{\mu}\cdot\vec{E}$. The above discussion suggests the possibility of looking for the effect on fluxons in two-dimensional superconductors.

We end by remarking that for magnetic monopoles the Lagrangian of Eqs. (4) and (12) may be derived[5,6] by using the dual form of Maxwell's equations.

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[a]Also at University of South Carolina, Columbia, S.C. 29208.

[1]Y. Aharonov and D. Bohm, Phys. Rev. **115**, 485 (1959).

[2]We neglect here effects due to electric moments induced by the external field (see below).

[3]R. Brout and F. Englert, Phys. Rev. Lett. **13**, 321 (1964); P. Higgs, Phys. Lett. **12**, 132 (1964), and Phys. Rev. Lett. **13**, 508 (1964).

[4]H. B. Nielsen and P. Olesen, Nucl. Phys. **B61**, 45 (1973).

[5]S. Coleman, in *The Unity of the Fundamental Interactions,* edited by Antonino Zichichi, Proceedings of the Nineteenth Course of the International School of Subnuclear Physics (Plenum, New York, 1983).

[6]F. Englert and P. Windey, Nucl. Phys. **B135**, 529 (1978).