# Equivalence of Cellular Automata to Ising Models and Directed Percolation

Eytan Domany[a]

Department of Applied Physics, Stanford University, Stanford, California 94305

and

Wolfgang Kinzel

Institut für Festkörperforschung der Kernförshungsanlage Jülich GmbH, D-5170 Jülich, West Germany

Time development of cellular automata in $d$ dimensions is mapped onto equilibrium statistical mechanics of Ising models in $d+1$ dimensions. Directed percolation is equivalent to a cellular automaton, and thus to an Ising model. For a particular case of directed percolation we find $\nu_{\parallel} = 2$, $\nu_{\perp} = 1$, $\eta_{\perp} = 0$.

In a most interesting recent article, Wolfram[1] has presented a large body of phenomenological observations and analytic results on the time development of cellular automata[2] (CA). CA provide extremely simple models for a variety of problems in physics,[3] e.g., nonequilibrium stochastic processes, self-organization, crystal-growth models,[4] chemistry (e.g., reaction models[5]), and biology.[6] CA may also become of some use in computer science.[1,7]

The class of CA studied by Wolfram constitute the deterministic limit of a more general class of stochastic CA. $d$-dimensional stochastic CA were studied under the name of crystal-growth models.[4] Their time development is equivalent to the equilibrium statistical mechanics of $(d+1)$-dimensional Ising models.[8] This exact mapping is presented here for the simple case of one-dimensional peripheral CA (PCA). We then show that the widely studied problem of directed percolation[9] (DP) in $d=2$ can be easily recast[10] as a particular stochastic PCA. Thus the DP problem is equivalent to an equilibrium Ising model. The standard problems of site and bond DP constitute particular cases of a more general class of DP problems. A special case of this more general class is solved exactly by mapping it onto a random-walk process.

*Cellular automata: Definitions.*—Consider a linear chain or ring of sites $i,j$; with each site associate a binary variable $v_i = 0, 1$. The "state" $V_t$ of the system at time $t$ is specified by the set of $v_{i,t}$. At odd (even) times, odd- (even-) indexed sites may change their state, according to preassigned probabilistic rules, and even- (odd-) indexed sites stay in the same state. Thus the full space-time history of our CA can be presented on a two-dimensional lattice. One may get $v_{i,t+1} = 0$ or 1, according to the conditional probabilities $P(v_{i,t+1} | v_{i-1,t}, v_{i+1,t})$ defined as follows:

$$P(1|0,0) = x, \quad P(1|1,1) = z, \quad P(1|0,1) = y. \quad (1)$$

Together with $P(0|v,v') = 1 - P(1|v,v')$, these rules define PCA. In general, $0 \leq x,y,z \leq 1$; when only $x,y,z = 0, 1$ are allowed, we obtain deterministic CA (DCA).

*Mapping of CA onto Ising models.*—Any space-time development of a CA, characterized by a set of values $v_{i,t} = V$, occurs with probability $P(V)$. Viewing the set of space-time indices $(i,t)$ as a regular $d=2$ lattice, we show that for any CA rules (i.e., $x,y,z$) there exists an Ising Hamiltonian $H(V)$ such that

$$P(V) = \exp[-H(V)] \quad (2)$$

for all $V$. If so, all space-time correlation functions and statistical averages of our CA can be expressed as an equilibrium average property of an appropriate Ising model.[8] We have

$$P(V) = P(V_1|V_0)P(V_2|V_1) \cdots P(V_{t+1}|V_t), \quad (3)$$

where $P(V_{t+1}|V_t)$ is the conditional probability to find the CA in state $V_{t+1}$ at time $t+1$, given it was in state $V_t$ at time $t$. However, because of the local nature of the CA rules,

$$P(V_{t+1}|V_t) = \prod_i{}' P(v_{i,t+1}|v_{i-1,t}, v_{i+1,t}), \quad (4)$$

where the prime indicates that for even (odd) times $i$ runs over even- (odd-) indexed sites. For the local $P(u|v,v')$ of Eq. (1), we identify $H(V)$ as an Ising Hamiltonian on a triangular lattice, with

$$-H = B\sum_k v_k + J\sum_{\langle kl \rangle}{}^{(-)} v_k v_l$$
$$+ D\sum_{\langle kl \rangle}{}^{(\backslash)} v_k v_l + E\sum_{\langle klm \rangle}{}^{(\nabla)} v_k v_l v_m, \quad (5)$$

where $\sum^{(-)}$ denotes sum over all horizontal bonds, $\sum^{(\backslash)}$ all diagonal bonds, and $\sum^{(\nabla)}$ sums over all *down-pointing* triangles (see Fig. 1), with the cou-
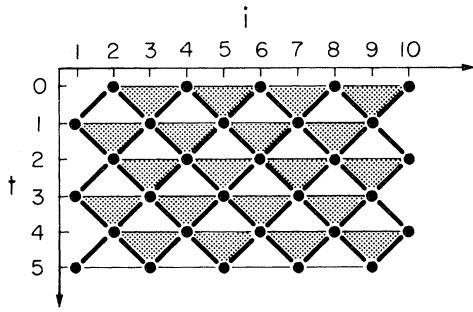
FIG. 1. Triangular Ising lattice, whose equilibrium-statistical mechanics describes the time evolution of a peripheral cellular automaton. Horizontal bonds $J$ (thin lines), diagonal bonds $D$ (heavy lines), three-site interactions $E$ (shaded down-pointing triangles), and field $B$ parametrize the Ising Hamiltonian.

plings given by $A = 1 - x$,

$$e^B = \frac{x(1-y)^2}{(1-x)^3}, \quad e^J = \frac{(1-x)(1-z)}{(1-y)^2},$$

$$e^D = \frac{y(1-x)}{x(1-y)}, \quad e^E = \frac{zx(1-y)^2}{y^2(1-x)(1-z)}. \quad (6)$$

Alternatively, by expressing $v_k = (1 + S_k)/2$, with $S_k = \pm 1$, we obtain a Hamiltonian similar to (5), with couplings $\tilde{B}$, $\tilde{J}$, $\tilde{D}$, and $\tilde{E}$. Thus, time development according to all possible PCA rules is described by equilibrium statistical mechanics of an Ising lattice gas on a triangular lattice.[8]

*Time development of DCA.*—The eight DCA rules correspond to various manners in which the zero-temperature limit of our Ising model can be taken. Time developments of the DCA correspond to the ground-state configurations of our Ising model, consistent with some preassigned configuration at one of the boundaries. When the ground state of $H$ is nondegenerate (or has finite degeneracy), it will be reached eventually and the DCA time development will correspond to the ground state. However, if the ground state is highly degenerate, complicated time development of the DCA is expected; the system picks one of a multitude of ground states that is consistent with the boundary condition. Thus various properties of DCA can be expressed in terms of correlations at $T = 0$ in degenerate Ising systems.[11]

*Phase transitions: Special subspaces.*—The point $x = y = z = \frac{1}{2}$ corresponds to $T = \infty$. We did not find phase transitions on any line that connects this point to any DCA point. Moreover, on the surface defined by vanishing three-spin coupling, i.e., $E = 0$, given by $zx/(1-z)(1-x) = y^2/(1-y)^2$, the problem has been treated by Verhagen,[8] and no

transition is found at any finite temperature. If we further impose $\tilde{B} = 0$, the subspace with $S \rightarrow -S$ Ising symmetry, given by $x = 1 - z$, $y = 1 - y = \frac{1}{2}$, is obtained. This line is precisely the disorder line[8,11] of the more general triangular Ising model with nearest-neighbor couplings $J$ and $\tilde{D}$; for this model Stephenson has calculated correlation functions in various directions and found diverging correlation length only as $T \rightarrow 0$ (i.e., $x \rightarrow 0$).[11] We did find phase transitions on the $x = 0$ (and by symmetry, the $z = 1$) surface of our cube.

*Directed percolation.*—Consider the square lattice of Fig. 1; any site may be present (probability $p$) or absent $(1 - p)$; any bond may be present (probability $q$) or absent $(1 - q)$. Assume that a single site is "wet" at $t = 0$; present bonds conduct "water" only in the downward (increasing $t$) direction, and only when both sites, connected by the bond, are present. For $q = 1$ this is the site-DP problem, and for $p = 1$ the bond-DP problem.[9] One asks, for given $p, q$ what is the probability of finding wet sites at level (time) $t$? To reduce this problem to CA rules,[10] note that whether any site $(i, t)$ is wet $(u_{i,t} = 1)$ or dry $(u_{i,t} = 0)$ depends only on the state of the down-pointing triangle whose bottom corner is $(i, t)$, with the CA rules $x = 0$, $y = pq$, $z = pq(2 - q)$. Thus the $x = 0$ plane of our CA is a generalized DP problem, in which the standard cases are embedded. Thus we have mapped the DP problem onto an Ising problem on the triangular lattice. The constraint $x = 0$ excludes those Ising configurations in which any down-pointing triangle has $u = 0$ (or $S = -1$) on both of its upper sites, and $u = 1$ (or $S = +1$) on its bottom. Within the space-allowed configurations we have (for bond DP) the Hamiltonian parametrized by $\exp(8\tilde{H}) = q^3(2 - q)$, $\exp(8\tilde{J}) = (2 - q)/q$, $\exp(8\tilde{D}) = q(2 - q)/(1 - q)^2$, and $\exp(8\tilde{E}) = (2 - q)/q$. The transition line was determined numerically.

The correlation length $\xi(y, z, N)$ is calculated exactly[12] for an infinite strip of finite width $N$. The transition point $(y_c, z_c)$ and the critical exponents $\nu_{\parallel}$ and $\nu_{\perp}$ in "time" and "space" directions are found using the scaling relation

$$\xi(t, h, 1/N) = b^{\nu_{\parallel}/\nu_{\perp}} \xi(b^{1/\nu_{\perp}} t, b^{\omega/\nu_{\perp}} h, b/N),$$

where $t$ is a scaling field in the $y$-$z$ plane and $b$ is the change of length scale [usually $b = N/(N-1)$]. We added a symmetry-breaking scaling field $h$ which scales with a new exponent $\omega$. From the usual (anisotropic) scaling relations[9] $\omega$, $\nu_{\parallel}$, and $\nu_{\perp}$ give the other critical exponents, $\beta = \nu_{\parallel} + \nu_{\perp} - \omega$, $\gamma = 2\omega - \nu_{\parallel} - \nu_{\perp}$. The full line of Fig. 2 gives $z_c(y)$ from $N = 10$, 11, and 12. For the upper part of the
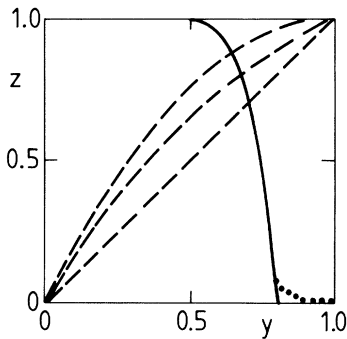
FIG. 2. The $x = 0$ plane of cellular automata rules corresponds to generalized directed percolation. Dashed lines correspond to bond, mixed bond-site $(q = p)$, and site percolation. The system percolates to the right of the (solid) transition line. On the $z = 1$ line the model is mapped onto a random-walk problem and solved.
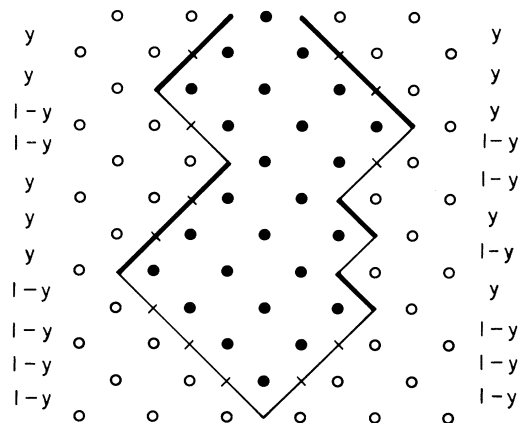


FIG. 3. A typical configuration of a wet (solid dots) domain, for the directed percolation model parametrized by $x = 0$, $z = 1$. The domain boundaries execute a random walk; the weight of each step is indicated on the right (for the right-hand boundary) and on the left (for the left-hand boundary).

phase boundary our results show only little $N$ dependence and can reliably be extrapolated to $N \to \infty$. However, near the $(z = 0, y = 1)$ point irregular variation of $z_c(y)$ with $N$ was observed, reflecting the fact that strips with different widths show very different reaction patterns[1] at the $(z = 0, y = 1)$ point, which results in an irregular function $\xi(N)$. The exponents $\nu_\parallel = 1.734 \pm 0.002$ and $\nu_\perp = 1.100 \pm 0.005$ were obtained for all three percolation problems shown in Fig. 2, indicating that the result is universal along the whole phase boundary (except the end points). To calculate the "magnetic" exponent $\omega$, we have identified $h$ with the probability $x$, since (i) $x$ destroys the transition as we have seen in our calculations, and (ii) $x$ is a source of $S = 1$ states and thus acts in the percolation language as a ghost field[13] which in usual percolation gives the critical symmetry-breaking scaling field $h$. At the critical point on the line $y = z$ we obtain $\omega = 2.57 \pm 0.03$, from which we get $\beta = 0.26 \pm 0.03$ and $\gamma = 2.21 \pm 0.05$ in agreement with previous results.[9]

On the line $x = 0, z = 1$ the model can be solved exactly, and we do find a DP transition as $y$ is varied, which occurs at the point with Ising symmetry, i.e., $y = \frac{1}{2}$. Say at $t = 0$ a single site has $u_{i,0} = 1$ while for all others $u_{j,0} = 0$. The probability of finding $u_{k,t} = 1$ for long times $t$ is given by $P_w(t) = \sum_U P(U) \rho_w(U,t)$, where $U$ is any configuration $u_{j,t}$ that appears with probability $P(U)$; $\rho_w(U,t) = 1$ if $U$ contains any site $(k,t)$ with $u_{k,t} = 1$, and $\rho_w(U,t) = 0$ otherwise. On the $x = 0, z = 1$ line only those configurations $U$ are allowed in which $no$ down-pointing triangle can have

either (i) $u = 0$ on both the upper corners and $u = 1$ on lowest corner, or (ii) $u = 1$ on both upper corners and $u = 0$ on the bottom. Configurations consistent with these constraints contain a single domain of "wet" sites in a sea of dry ones; furthermore, $no$ $branching$ of the wet domain is allowed (see Fig. 3).[14] The weight $P(U)$ of such a graph is determined by the right- and left-hand boundaries of the domain, each of which goes either right or left with increasing $t$, until they meet and "annihilate." The right-hand boundary carries a weight $y$ for each step to the right, and weight $1 - y$ for each step to the left. The reverse holds for the left-hand boundary. $P_w(t)$ is the sum of all graphs that have not terminated after $t$ steps; i.e., the probability that two random walkers with right/left transition probabilities defined above, starting at a separation of two steps, have not met before $t$ time steps. Instead of dealing with two random walkers, we can consider only the single random walk performed by their relative positions, or the difference walk. This walk is characterized by the transition probabilities $P(d \to d+1) = p_+ = y^2$, $P(d \to d) = p_0 = 2y(1 - y)$, $P(d \to d-1) = p_- = (1 - y)^2$, where $d$ is half the distance between the walkers. Thus $P_w(t)$ is given by the probability that a gamber with initial capital of 1, with probability $p_+$ $(p_-)$ of winning (losing) one unit and $p_0$ of maintaining his holdings, will $not$ reach his ruin (zero capital) before $t$ games.[15] We first calculate the probability of ruin after exactly $t$ steps, $r_t$, summing over all paths starting at 1 and reaching 0 $for$ $the$ $first$ $time$ at $t$. If

such a path has $\sigma$ steps with no change we have

$$r_t = \sum_{\sigma=0}^{t-1}{}' \binom{t-1}{\sigma} p_0^\sigma \tilde{r}_{t-\sigma},$$

where the prime indicates that $\sigma$ must have the same parity as $t-1$, and $\tilde{r}_k$ is given by[15]

$$\tilde{r}_k = \frac{1}{k} p_+^{(k-1)/2} p_-^{(k+1)/2} \binom{k}{(k+1)/2}.$$

Replacing the sum by an integral, and evaluating it by saddle-point integration, we find for $\epsilon = \frac{1}{2} - y \ll 1$ (note that $p_+ - p_- = -2\epsilon$) that $r_t \sim e^{-4t\epsilon^2}/t^{3/2}$ for $t \gg 1$. Therefore the correlation length in the time direction is $\xi_{\parallel} \sim \epsilon^{-2}$, and $\nu_{\parallel} = 2$. Also, since

$$P_w(t+1) = 1 - \sum_{t'=1}^{t} r_{t'} = P_w(t) - r_t,$$

we must have at criticality $(\epsilon = 0)$ $P_w(t) \sim t^{-1/2}$. To calculate correlations in the $\perp$ direction, note that the $x=1, z=0$ line is in the subspace of Verhagen's solution.[8] In his notation, on this line $b=0$ and $a = y(1-y) \simeq 1 - 4\epsilon + O(\epsilon^2)$. He found that $P(u_{l,t}=1|u_{0,t}=1) = a^l \simeq e^{-4\epsilon l}$ and therefore $\xi_{\perp} \sim 1/\epsilon$ and $\nu_{\perp} = 1$, while $\eta_{\perp} = 0$.

By combining the mapping of one-dimensional peripheral cellular automata onto Ising problems in two dimensions with representation of directed percolation as a particular case of PCA, we have recast the DP problem as an Ising model. A line of DP transitions was found numerically, and a specific DP model was solved exactly.

---

(a)Permanent address: Department of Electronics, Weizmann Institute of Science, Rehovot, Israel.

[1]S. Wolfram, Rev. Mod. Phys. **55**, 601 (1983). See also Physica (Utrecht) **10D**, 1 (1984).

[2]J. von Neumann, *Theory of Self Reproducing Automata* (University of Illinois, Urbana, 1966).

[3]P. Grassberger, Physica (Utrecht) **10D**, 52 (1984); E. Ott, Rev. Mod. Phys. **53**, 655 (1981); H. Haken, *Chaos and Order in Nature* (Springer, Berlin, 1981).

[4]T. R. Welberry and R. Galbraith, J. Appl. Crystallogr. **6**, 87 (1973); T. R. Welberry and G. H. Miller, Acta Crystallogr., Sec. A **34**, 120 (1978).

[5]G. Nicolis and I. Prigogine, *Self Organization in Nonequilibrium Systems* (Wiley, New York, 1977).

[6]S. Ulam, Annu. Rev. Biochem. **255**, (1974).

[7]B. A. Huberman and T. Hogg, Phys. Rev. Lett. **52**, 1048 (1984).

[8]A. M. W. Verhagen, J. Stat. Phys. **15**, 213 (1976); I. G. Enting, J. Phys. C **10**, 1379 (1977), and J. Phys. A **11**, 555, 2001 (1978).

[9]W. Kinzel, in *Percolation Structures and Processes, Annals of the Israel Physical Society, Vol. 5*, edited by G. Deutscher, R. Zallen, and J. Adler (Bar-Ilan University, Ramat-Gan, Israel, 1983), p. 425.

[10]D. Dhar, *Stochastic Processes, Formalism and Applications*, edited by G. S. Agarwal and S. Dattagupta (Springer, Berlin, 1983).

[11]J. Stephenson, J. Math. Phys. **5**, 1009 (1964), and **11**, 413, 420 (1970), and Phys. Rev. B **1**, 4405 (1970).

[12]W. Kinzel and J. Yeomans, J. Phys. A **14**, L163 (1981).

[13]R. B. Griffiths, J. Math. Phys. **8**, 484 (1967).

[14]Precisely these graphs contribute to the Mauldon process $B_l$; however, the weights of each graph are different in our model from those assigned by J. G. Mauldon, in *Proceedings of the Fourth Berkeley Symposium on Mathmatics, Statistics and Probability*, edited by Jerzy Neyman (Univ. of California, Berkeley, 1961), Vol. 2, p. 337.

[15]W. Feller, *An Introduction to Probability Theory and its Applications* (Wiley, New York, 1968).