Self-Consistency Effects in Quasilinear Theory: A Model for Turbulent Trapping

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Model equations are proposed in an attempt to take into account the self-consistency effects which invalidate the quasilinear theory in one dimension. The diffusion coefficient and mode growth rate are found to be increased as compared with their quasilinear predictions.

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In this Letter we propose model equations in an attempt to take into account the mode-coupling terms which originate in the self-consistency of the electric field and which invalidate the quasilinear theory in one dimension. We restrict ourselves to situations of weak turbulence that we define as follows: The nonlinear resonance width in frequency space is negligible so that one may assign a well-defined frequency ω_k to a given wave number k; in this case the validity conditions for quasilinear theory were originally thought[1] to reduce to those for Fokker-Planck (FP) equations, namely $\epsilon \equiv \max(\tau_c/\tau_f,\ \tau_c/\tau_E) << 1$, where τ_f and τ_E are the characteristic times for nonlinear evolution of the average distribution function and of the spectral density; the quantity τ_c denotes the correlation time of the two-point correlation function of the electric field as seen by a resonant particle, namely $\tau_c \simeq 1/\Delta(\omega_k - kv) \simeq 1/\Delta k\,|v_p - v_g|$, where Δk is the width in k space of the unstable waves, v_p and v_g are the phase and group velocities of a characteristic unstable mode. However, it has been recently proved[2] that the applicability of quasilinear theory is restricted by an additional constraint of small-field amplitudes, namely $(k^2 D^{ql})^{1/3} << \gamma_k^{ql}$ where D^{ql} and γ_k^{ql} are the quasilinear diffusion coefficient and mode growth rate; the latter condition represents a drastic constraint upon the turbulence level. In order to summarize simply why the quasilinear theory loses its validity in the domain $(k^2 D^{ql})^{1/3} > \gamma_k^{ql}$, we define in the first part of the Letter what should be a self-consistent FP description of the Vlasov-Poisson turbulence; we then show that the quasilinear theory does not respect some constraints imposed by the self-consistency of the electric field; lastly we introduce our turbulent trapping model as a self-consistent FP description of the Vlasov-Poisson turbulence.

In FP theory the averaged distribution function $\langle f_1 \rangle = \langle f(x_1, v_1, t)\rangle$ evolves according to an equation of the form

$$\partial_t \langle f_1 \rangle = \partial_{v_1} D(v_1) : \partial_{v_1} \langle f_1 \rangle, \qquad (1)$$

with $D(v_1) = \frac{1}{2}\langle \Delta v_1^2 / \Delta t\rangle$ where the velocity increase $\Delta v_1 \equiv v_1(t+\Delta t) - v_1(t)$ is computed by integrating the equations of the particles' motion in which the electric field is considered to be a stochastic process. Similarly, the evolution of the two-point correlation function

$$\langle f_1 f_2 \rangle \equiv \langle f(x_1, v_1, t) f(x_2, v_2, t)\rangle$$

is given by a FP equation of the form

$$(\partial_t + v_- \partial_{x_-})\langle f_1 f_2\rangle = \sum_{(i,j)=(1,2)} \partial_{v_i} D_{ij} \partial_{v_j} \langle f_1 f_2 \rangle, \qquad (2)$$

with

$$D_{ij}(v_i, v_j, x_i - x_j) = \tfrac{1}{2}\langle \Delta v_i \Delta v_j / \Delta t\rangle;$$

from the latter definition there follow the relations $D_{ii} = D(v_i)$ and $D_{12} = D_{21} \to D(v_i)$ for $(x_1 - x_2,\ v_1 - v_2) \to 0$. By subtracting Eq. (2) from Eq. (1), one obtains the equation of evolution for the irreducible correlation function $\langle \delta f_1 \delta f_2 \rangle = \langle f_1 f_2 \rangle - \langle f_1 \rangle \langle f_2 \rangle$. It reads

$$(\partial_t + T_{12})\langle \delta f_1 \delta f_2 \rangle = S_{12} \qquad (3)$$

with $T_{12} \equiv v_- \partial x_- - \sum_{i,j} \partial_{v_i} D_{ij} \partial_{v_j}$ and

$$S_{12} \equiv [\partial_{v_1} D_{12} \partial_{v_2} + 1 \leftrightarrow 2]\langle f_1 \rangle \langle f_2 \rangle;$$

x_- and v_- denote the relative coordinates $x_- = x_1 - x_2$ and $v_- = v_1 - v_2$.

Let us now derive the constraints upon the diffusion coefficients which are imposed by the self-consistency of the electric field. First of all, the averaging of the Vlasov equation yields the following relation:

$$D(v_1)\partial_{v_1}\langle f_1 \rangle = -(q/m)\langle E(1)\delta f(1)\rangle \qquad (4)$$

in which the correlation function $\langle E(1)\delta f(1)\rangle$ is self-consistently computed by using the Poisson equation as follows:

$$\langle \delta f(j) \partial_{x_i} E(i)\rangle = (q/\epsilon_0)\int dv_i \langle \delta f(j) \delta f(i)\rangle. \qquad (5)$$

The constraint (4) upon $D(v)$ follows directly from the Vlasov equation, i.e., from the conservation of the distribution function $f(x,v,t)$ along the particles' orbits. On the other hand, as shown by Boutros-Ghali and Dupree,[3] an analogous expression may also be derived for D_{12} if one takes into account the conservation of f^2 along the particles' orbits; from this latter property it follows indeed that in the limit $(x_-, v_-) \to 0$ one has exactly

$$\partial_t \langle \delta f(1) \delta f(2) \rangle \to -(q/m)[\langle E(1)\delta f(2)\rangle + 1 \leftrightarrow 2]\partial_{v_+}\langle f \rangle,$$

with $v_+ = (v_1+v_2)/2$; in this same limit the operator T_{12} vanishes so that Eq. (3) yields

$$\partial_t \langle \delta f(1)\delta f(2)\rangle \to 2D_{12}(v_+, v_+, x_-)(\partial_{v_+}\langle f \rangle)^2.$$

By matching the two limits, one immediately obtains a second relation upon the diffusion coefficients, namely

$$D_{ij}(v_+, v_+, x_-)\partial_{v_+}\langle f\rangle = -(q/m)\langle E(i)\delta f(j)\rangle, \qquad (6)$$

in which $\langle E(i)\delta f(j)\rangle$ is computed self-consistently by use of Eq. (5). On the other hand, the mode growth rate γ_k is provided by the dispersion relation obtained by writing the Poisson equation in the form

$$\partial_{x_1}\partial x_2 \langle E(1)E(2)\rangle = (q/\epsilon_0)^2 \int dv_1\, dv_2 \langle \delta f(1)\delta f(2)\rangle;$$

the conservation of energy might then be written as a third condition relating the diffusion coefficient $D(v)$ to the growth rate γ_k. We may therefore conclude that the self-consistency of the electric field is properly taken into account in a FP description of the Vlasov-Poisson turbulence only if at least three constraints upon the diffusion coefficients are fulfilled, as given, namely, by Eqs. (5) and (6) and by energy conservation.

We now summarize the proof of inconsistency of the quasilinear theory in the regime $(k^2 D^{ql})^{1/3} \gg \gamma_k^{ql}$ as given by Adam, Laval, and Pesme.[4] As is well known, the quasilinear approximation consists in neglecting all the mode-coupling terms in the equation of evolution of the spectral density. In the Fokker-Planck description this approximation is equivalent to the assumption that the electric field behaves as a Gaussian process along the free orbits when one computes the average $\langle \Delta v^2 \rangle$; this assumption has received the name of "quasi-Gaussian" *Ansatz*[5] and yields[4]

$$D(v) = D^{ql}(v), \qquad (7a)$$

$$D_{12} = D(v_+)\cos k_+(x_1 - x_2). \qquad (7b)$$

Equation (7b) is valid in the limit of small separation in v space, $|v_-| \ll [D^{ql}(v_+)/k_+]^{1/3}$ where k_+ denotes the wave number corresponding to resonant interaction with the particles of velocities $v_1 \sim v_2 \sim v_+$; k_+ is namely defined to be the solution of the equation $\omega_{k_+} = k_+ v_+$. The validity of Eq. (7b) is also restricted by a condition upon separation in x space, $|x_-| \ll l_{tt}$, with $l_{tt} = |v_+ - v_{g+}|(k_+^2 D^{ql})^{-1/3}$, where $v_{g+} = (\partial\omega_k/\partial k_+)$; the quantity l_{tt} appears later on to play the role of a characteristic length for correlation enhancement. By using D_{12} as given by Eq. (7b) one may exactly solve Eq. (3) and compute the correlation function $\langle \delta f_1 \delta f_2\rangle$; inserting the latter into the dispersion relation, one finds $\gamma_k = \gamma_k^{ql}$ for $(k^2 D^{ql})^{1/3} \ll \gamma_k^{ql}$ and $\gamma_k = (A_{tt})^{1/2}\gamma_k^{ql}$ in the opposite regime[6]; A_{tt} is a numerical constant larger than unity which is given later. Since the latter result does not respect energy conservation, the authors of Ref. 4 concluded in favor of a basic inconsistency of the quasilinear theory, which can be rephrased as follows: The zero-order modification of the growth rate demonstrates the existence of mode-coupling terms which yield a contribution of the same order of magnitude as the quasilinear term; on the other hand these nonnegligible mode-coupling terms contradict *a posteriori* the initial assumption of statistics close to Gaussian, thus invalidating the quasilinear values of D and D_{12} as given by Eqs. (7). Let us stress that a standard propagator renormalization cannot solve the problem since one may explicitly show[2,5] that there exists in the regime $(k^2 D^{ql})^{1/3} \gg \gamma_k^{ql}$ an infinite class of mode-coupling terms which cannot be properly taken into account by a propagator renormalization; these mode-coupling terms are associated with non-Gaussian effects[2] which originate in the self-consistency of the electric field. Similar conclusions concerning the inadequacy of the quasi-Gaussian theories (or test-field theory) for describing the Vlasov turbulence have been reached by DuBois[7] from a formal analysis based upon Kraichnan's[8] direct-interaction approximation (DIA); the interested reader is referred to the work of Dubois and Espedal and Krommes and Kleva[9] for an introduction to DIA for Vlasov-Poisson turbulence.

The rest of our Letter is now devoted to the regime where the quasilinear theory breaks down, i.e., $(k^2 D^{ql})^{1/3} \gg \gamma_k^{ql}$. Let us first display the two

Ansätze which underly our model:

(i) We assume that the non-Gaussian deviations do not invalidate the FP description of $\langle f_1 \rangle$ and $\langle f_1 f_2 \rangle$. Concerning the evolution of $\langle f_1 \rangle$ this assumption may be justified as follows: Dimensional analysis shows easily[2] that because of the self-consistency imposed by the Poisson equation, the statistical properties of the electric field are characterized in addition to τ_c by a second correlation time τ_c^{eff} of the order of

$$\tau_c^{\text{eff}} \simeq \min[(\gamma_k^{ql})^{-1}, (k^2 D^{ql})^{-1/3}];$$

similar results have been found by various authors.[10] In the regime $(k^2 D^{ql})^{1/3} >> \gamma_k^{ql}$, the inequality $\tau_c^{\text{eff}}/\tau_f << 1$ is satisfied so that the mode-coupling terms invalidate only the quasilinear approximation for the diffusion coefficient $D(v)$ and do not bring into question the Fokker-Planck description itself. On the other hand, as concerns the evolution of the two-point correlation function $\langle f_1 f_2 \rangle$, one cannot justify rigorously the use of the FP equation (2) because the characteristic time of evolution for $\langle f_1 f_2 \rangle$ is of the same order as τ_c^{eff}; our first *Ansatz* consists thus in assuming that the evolution of $\langle f_1 f_2 \rangle$ may be correctly described by a FP equation.

(ii) Our second *Ansatz* does not receive any rigorous demonstration either but follows the usual ideas of renormalization techniques: We indeed assume that the effect of the mode-coupling terms is to renormalize both $D(v)$ and D_{12} as compared with their quasilinear values (7) in such a way that the relation (7b) still holds; namely, we assume that $D_{12} = D(v_+) \cos k_+(x_1 - x_2)$, in which the diffusion coefficient $D(v)$ represents now the sum of the quasilinear contribution and of the mode-coupling terms. A more detailed analysis shows that this latter *Ansatz,* and therefore our model, are meaningful only if the waves are weakly dispersive as prescribed by an inequality[2,7] that we assume henceforth, namely

$$(\partial^2 \omega_k / \partial k^2)[(k^2 D^{ql})^{1/3}/|v_p - v_g|]^2 << \gamma_k^{ql};$$

in the case of the weak warm-beam instability, this inequality is fulfilled for almost all the nonlinear evolution.

We now yield the solution of the turbulent-trapping model in its regime of applicability previously defined; first of all, the correlation function $\langle \delta f(1) \delta f(2) \rangle$ may be exactly computed in terms of the unknown diffusion coefficient $D(v)$ by solving Eq. (3); one obtains

$$\langle \delta f(1) \delta f(2) \rangle = \tau_{tt}(x_-, v_-) 2D(v_+)(\partial_{v_+} \langle f \rangle)^2$$

with

$$\tau_{tt}(x_-, v_-) = [2k_+^2 D(v_+)]^{-1/3} \sum_{p=-\infty}^{+\infty} H_p(v_-) \exp(ipk_+ x_-) \tag{8}$$

with

$$H_p(v_-) = \sum_{n=1}^{\infty} J_{p-n}(-n)[J_n(n)/n] \int_{-\infty}^{0} \exp(i\mu V) \exp(\mu^3/3n)\, d\mu - \sum_{n=-\infty}^{-1} J_{p-n}(-n)[J_n(n)/n] \int_0^{\infty} \exp(i\mu V) \exp(\mu^3/3n)\, d\mu,$$

where $V = v_-[2D(v_+)/k_+]^{-1/3}$ and where $J_n(x)$ denotes the Bessel function of order n. The solution (8) is valid for $|v_-| << [D(v_+)/k_+]^{1/3}$ and $|x_-| << l_{tt}$; otherwise the resonance broadening predictions apply. The quantity $\tau_{tt}(x_-, v_-)$ plays a role exactly similar to the usual clumping time[5] $\tau_{\text{cl}}(x_-, v_-)$ of the strong-turbulence case, i.e., the case where the nonlinear width $\delta\omega_k$ is of the same order as ω_k; in particular, in the limit $(x_-, v_-) \to 0$, $\tau_{tt}(x_-, v_-)$ exhibits the same logarithmic divergence as $\tau_{\text{cl}}(x_-, v_-)$; in our case, however, because of the hypothesis of weak turbulence, $\tau_{tt}(x_-, v_-)$ is a periodic function of x_- (in the limit $x_- << l_{tt}$) so that $\tau_{tt}(x_-, v_-)$ exhibits this characteristic divergence whenever the relative coordinate x_- fulfills $x_- \simeq n 2\pi/k_+$ and $|x_-| << l_{tt}$, where n is an integer. This result may receive the following physical interpretation: A particle with velocity v interacts resonantly with those modes whose wave numbers k fulfill $|k - k(v)| \lesssim [k(v)^2 D(v)]^{1/3}/|v|$; these modes form a wave packet characterized in the particle moving frame by a correlation length of the order of l_{tt}; therefore the motions of two particles with velocities v_1 and v_2 are strongly correlated as long as their relative distance in phase space is such that $|v_-| \lesssim [D(v_+)/k_+]^{1/3}$ and $|x_-| \lesssim l_{tt}$. This picture and the harmonic generation described by Eq. (9) are very reminiscent of the trapping situation[11] in which a wave packet would replace a monochromatic wave; this remark justifies the terminology "turbulent trapping" that we use to describe our model.

Inserting now $\langle \delta f(1) \delta f(2) \rangle$ into Eq. (5) and using Eq. (4) one obtains $D(v_p)/D^{ql}(v_p) = \gamma_k/\gamma_k^{ql}$; on the other hand the growth rate γ_k is provided by the dispersion relation which yields

$$(\gamma_k/\gamma_k^{ql})^2 = A_{tt} D(v_p)/D^{ql}(v_p),$$

where A_{tt} is a numerical constant given just below. Combining the two latter results, one obtains the values of the renormalized diffusion coefficient D and growth rate γ_k, namely

$$D/D^{ql} = \gamma_k/\gamma_k^{ql} = A_{tt} \equiv 1 + \{8 \sum_{n=1}^{\infty} [J_n^2(n)/n] - 2\}; \qquad (9)$$

one finds numerically $A_{tt} \simeq 2.2$; the unit term in A_{tt} stands for the quasilinear contribution and the factor in the braces represents the mode-coupling contribution; Eq. (9) preserves energy conservation and one may also check *a posteriori* that the self-consistency condition (6) is satisfied. For completeness we may say that in the regime $(k^2 D^{ql})^{1/3} << \gamma_k^{ql}$, our model leads to the same equation as Eq. (9) in which A_{tt} reduces to unity, so that one recovers the quasilinear theory as expected.

Pesme and Dubois have shown[12] that the turbulent-trapping model differs in detail from the DIA predictions for this problem by omitting most of the characteristic polarization terms of the DIA; they conjecture that nevertheless the turbulent-trapping model, although predicting different correlation functions, *might* yield in the limit $\epsilon \rightarrow 0$ the same predictions as the DIA for the diffusion coefficient and growth rate. Although not yet justified, our model has, however, the merit of yielding an explicit and consistent prediction for a net increase of energy transfer between the waves and the particles where the increasing factor A_{tt} is in a qualitatively good agreement with preliminary numerical results.[13]

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