Landau-Lifshitz Equation of Ferromagnetism: Exact Treatment of the Gilbert Damping

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In the Landau-Lifshitz equation which describes the evolution of spin fields in nonequilibrium continuum ferromagnets, by stereographic projection of the unit sphere of spin onto a complex plane, it is shown that the effect of the Landau-Lifshitz-Gilbert damping term is a mere rescaling of time by a complex constant. Consequently, for any given undamped motion of spatially regular and/or irregular spin structures, the nature of the damping can be analyzed exactly in a simplified manner.

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The Landau-Lifshitz (LL) equation[1] which describes the evolution of spin fields in continuum ferromagnets bears a fundamental role in the understanding of nonequilibrium magnetism,[2,3] just as the Navier-Stokes equation does in that of fluid dynamics. In the context of nonlinear dynamics, it is being realized that the LL equation possesses fascinating geometrical properties and that its special versions without damping in $(1+1)$ dimensions are completely integrable soliton systems.[4-7] Here we wish to show the astonishing fact that the effect of Landau-Lifshitz-Gilbert or simply the Gilbert damping[1-3] is just a rescaling of the time variable t by a complex constant, so that for every given solution of the undamped LL equation in any dimension the exact solution of the fully damped version can be given straightforwardly. The way in which we demonstrate this result is by projecting the unit sphere of spin $\vec{S}(\vec{r},t)$ stereographically onto a complex plane of $\omega(\vec{r},t)$ and then rewriting the LL equation in terms of the latter variable.

It is well known that the normalized form of the Landau-Lifshitz equation for the ferromagnetic spin system is of the form[1,2]

$$\partial\vec{S}(\vec{r},t)/\partial t=\vec{S}\times\vec{F}_{\text{eff}}-\lambda\vec{S}\times(\vec{S}\times\vec{F}_{\text{eff}})=\vec{S}\times\vec{F}_{\text{eff}}+\lambda[\vec{F}_{\text{eff}}-(\vec{S}\cdot\vec{F}_{\text{eff}})\vec{S}], \tag{1}$$

where $\vec{S}\equiv(S^x, S^y, S^z)$ and $\vec{S}^2=1$, and λ is a dimensionless Gilbert damping parameter. In Eq. (1) the effective field $\vec{F}_{\text{eff}}$ typically contains contributions from exchange interaction, crystalline anisotropy, magnetostatic self-energy, external magnetic fields, thermal fluctuations, etc. Equation (1) can also be written in the alternative Gilbert form as

$$\partial\vec{S}(\vec{r},t)/\partial t=\vec{S}\times\vec{F}_{\text{eff}}-\lambda(\vec{S}\times\partial\vec{S}/\partial t). \tag{2}$$

One can easily check that Eqs. (1) and (2) are identical to within a constant scaling factor $1+\lambda^2$ of the time variable t. We will use both the forms in Eqs. (1) and (2) in the following analysis.

To begin with we consider a typical form of $\vec{F}_{\text{eff}}$, corresponding to a uniaxial anisotropic Heisenberg ferromagnet in external magnetic fields:

$$\vec{F}_{\text{eff}}=\nabla^2\vec{S}-2A(\vec{S}\cdot\vec{n})\vec{n}+\mu\vec{B},\quad \vec{n}=(0,0,1), \tag{3}$$

where A is the anisotropy parameter ($A>0$, easy plane; $A<0$, easy axis), μ is the gyromagnetic ratio in Bohr magnetons, and $\vec{B}=\vec{B}(t)$ is the external magnetic field. It may be noted that in the undamped case Eq. (1) or (2) for Eq. (3) may be derived starting from a field Hamiltonian

$$H=\int d^3r\,\tfrac{1}{2}[(\nabla\vec{S})^2+2A(\vec{S}\cdot\vec{n})^2-2\mu\vec{B}\cdot\vec{S}],$$

with suitable Poisson brackets for the spin fields.[2,4,8]

In the undamped case ($\lambda=0$) recent investigations have established that the $(1+1)$-dimensional version of the LL equation for the pure isotropic case [$A=0$ and $\vec{B}=0$ in Eq. (3)] is completely integrable and is equivalent to a nonlinear Schrödinger equation,[4] and that the elementary excitations are envelope solitons and magnons. Also for $A\neq 0$[5,6] and $\vec{B}=(0,0,B^L)$ it is an integrable solitonic system. We wish to consider now the effect of the damping terms proportional to λ in Eq.

(1) or (2) on the undamped spin motion. Traditional treatments of Eq. (1) in polar coordinates tend to mix up the evolutions of the two angles in a complicated way[3] and so the damping is treated only approximately. Even the other geometrical parametrizations which proved to be successful in the undamped case tend to complicate the treatment[8] of damping. In the following, however, we show that the parametrization of the spin field in terms of a stereographic variable simplifies the structure of Eq. (1) or Eq. (2) drastically.

We therefore project the unit sphere of spin $\vec{S}^2(\vec{r},t)=1$ stereographically onto a complex variable $\omega(\vec{r},t)$[9]:

$$\omega(\vec{r},t)=\frac{S^x+iS^y}{(1+S^z)}, \tag{4a}$$

$$S^x(\vec{r},t)=\frac{2\,\mathrm{Re}\,\omega(\vec{r},t)}{(1+\omega\omega^*)},\quad S^y(\vec{r},t)=\frac{2\,\mathrm{Im}\,\omega(\vec{r},t)}{(1+\omega\omega^*)},\quad S^z(\vec{r},t)=\frac{(1-\omega\omega^*)}{(1+\omega\omega^*)}. \tag{4b}$$

Then the derivatives are easily seen to be

$$S_t^x=\partial S^x/\partial t=\Gamma[\omega_t(1-\omega^{*2})+\omega_t^*(1-\omega^2)], \tag{5a}$$

$$S_t^y=-i\Gamma[\omega_t(1+\omega^{*2})-\omega_t^*(1+\omega^2)], \tag{5b}$$

$$S_t^z=-2\Gamma(\omega_t\omega^*+\omega\omega_t^*), \tag{5c}$$

and

$$\nabla^2S^x=\Gamma[(1-\omega^{*2})\nabla^2\omega+(1-\omega^2)\nabla^2\omega^*]$$
$$-2\Gamma^{3/2}[2(\omega+\omega^*)\nabla\omega\cdot\nabla\omega^*+\omega^*(1-\omega^{*2})(\nabla\omega)^2+\omega(1-\omega^2)(\nabla\omega^*)^2], \tag{6a}$$

$$\nabla^2S^y=-i\Gamma[(1+\omega^{*2})\nabla^2\omega-(1+\omega^2)\nabla^2\omega^*]$$
$$+2i\Gamma^{3/2}[2(\omega-\omega^*)\nabla\omega\cdot\nabla\omega^*+\omega^*(1+\omega^{*2})(\nabla\omega)^2-\omega(1+\omega^2)(\nabla\omega^*)^2], \tag{6b}$$

$$\nabla^2S^z=-2\Gamma^{3/2}[\omega^*(1+\omega\omega^*)\nabla^2\omega+\omega(1+\omega\omega^*)\nabla^2\omega^*$$
$$+2(1-\omega\omega^*)\nabla\omega\cdot\nabla\omega^*-2\omega^{*2}(\nabla\omega)^2-2\omega^2(\nabla\omega^*)^2], \tag{6c}$$

where $\Gamma=(1+\omega\omega^*)^{-2}$. We then reexpress the individual components of Eq. (1) for the specific form of $\vec{F}_{\text{eff}}$ in Eq. (3). The x component of Eq. (1) with $\vec{B}=0$ (for $\vec{B}\neq 0$, see below) reads

$$\partial S^x/\partial t=(S^y\nabla^2S^z-S^z\nabla^2S^y)-2AS^yS^z+\lambda[\nabla^2S^x-\{\vec{S}\cdot\nabla^2\vec{S}-2A(\vec{S}\cdot\vec{n})^2\}S^x]. \tag{7}$$

By use of the stereographic transformation in Eqs. (4)–(6) and the fact that

$$\vec{S}\cdot\nabla^2\vec{S}=-4\nabla\omega\cdot\nabla\omega^*/(1+\omega\omega^*)^2, \tag{8}$$

Eq. (7) can be rewritten as

$$(1-\omega^{*2})G(\omega,\omega^*)-(1-\omega^2)G^*(\omega,\omega^*)=0, \tag{9}$$

where

$$G(\omega,\omega^*)=i(1+\omega\omega^*)\omega_t+(1-i\lambda)[(1+\omega\omega^*)\nabla^2\omega-2\omega^*(\nabla\omega)^2+2A\omega(1-\omega\omega^*)]. \tag{10}$$

Similarly the other two components of Eq. (1) become

$$-i(1+\omega^{*2})G(\omega,\omega^*)-i(1+\omega^2)G^*(\omega,\omega^*)=0 \tag{11}$$

and

$$2\omega^*G(\omega,\omega^*)-2\omega G^*(\omega,\omega^*)=0. \tag{12}$$

Consistency of Eqs. (9)–(12) then obviously implies $G(\omega,\omega^*)=0$ and $G^*(\omega,\omega^*)=0$ so that the evolution equation for the stereographic variable $\omega(\vec{r},t)$ in the presence of damping becomes

$$i(1+\omega\omega^*)\omega_t+(1-i\lambda)[(1+\omega\omega^*)\nabla^2\omega-2\omega^*(\nabla\omega)^2+2A\omega(1-\omega\omega^*)]=0, \tag{13}$$

and its complex conjugate. On the other hand, redefining the time variable

$$t \to \tau = (1 - i\lambda) t, \tag{14}$$

we obtain

$$i(1+\omega\omega^*)\omega_\tau + [(1+\omega\omega^*)\nabla^2\omega - 2\omega^*\nabla\omega\cdot\nabla\omega + 2A\omega(1-\omega\omega^*)] = 0, \tag{15}$$

which is exactly the same as the undamped evolution equation for ω endowed here with the scaled time τ. Thus for every solution in the $\lambda = 0$ case, we have the corresponding solution in the damped ($\lambda \neq 0$) case just with the rescaling in Eq. (14) of the time parameter. The corresponding damped spin field $\vec{S}(\vec{r},t)$ can then be constructed simply from Eq. (4).

It can be easily observed that the effect of magnetic fields $\mu\vec{B}$ on $\vec{F}_{\rm eff}$ in Eq. (3) does not alter the above fact either. We verify that the effect of a longitudinal field $\vec{B}(t) = (0,0,B^L(t))$ is to add a term $-\mu B^L(1+\omega\omega^*)\omega$ to the terms proportional to $(1-i\lambda)$ in Eq. (13) and the effect of a transverse field $\vec{B}(t) = (B^T(t),0,0)$ is to add a factor $\frac{1}{2}\mu B^T(1-i\lambda)(1+\omega\omega^*)(1-\omega^2)$ to the left-hand side of Eq. (13).

More generally, we claim that for a given arbitrary $\vec{F}_{\rm eff}$, our above assertion is true. The reason for this can be elucidated by manipulation of the Gilbert form in Eq. (2): The time derivative terms in Eq. (2) can be summarized in stereographic coordinates as

$$\partial\vec{S}/\partial t + \lambda(\vec{S}\times\partial\vec{S}/\partial t) = [(1+i\lambda)/(1+\omega\omega^*)^2][(1-\omega^{*2})\vec{e}_1 - i(1+\omega^{*2})\vec{e}_2 - 2\omega^*\vec{e}_3]\omega_t + \text{c.c.}, \tag{16}$$

where the unit orthonormal vectors $\vec{e}_i$, $i = 1,2,3$, define $\vec{S} = S^x\vec{e}_1 + S^y\vec{e}_2 + S^z\vec{e}_3$ and c.c. stands for complex conjugate in Eq. (16). Since the remaining term in Eq. (2), i.e., $\vec{S}\times\vec{F}_{\rm eff}$, includes no λ-dependent term, the spin evolution equation in Eq. (2) rewritten in stereographic coordinates is obtained from its undamped version by multiplying the time variable by the factor $(1+i\lambda)^{-1}$ in the latter. Combining this fact with the already existing scale difference $1+\lambda^2$ between Eqs. (1) and (2), we once again but in a more general way arrive at the earlier conclusion that the effect of Landau-Lifshitz-Gilbert damping is just a rescaling of time by the factor $1-i\lambda$ of the undamped spin motion. This simplification makes feasible a systematic study on pattern-forming transitions and kinetics of topological singularities in nonequilibrium magnets,[3,10] where the Gilbert damping as well as driving magnetic fields play an essential role.

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