Smoothing Out Spatially Closed Cosmologies

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We discuss a smoothing-out procedure that deforms a family of locally inhomogeneous and anisotropic spatially closed cosmological models into closed Friedmann-Robertson-Walker universes. This gives a precise content to the averaging hypothesis tacitly assumed in cosmology by providing explicitly the correction terms to the physical sources induced on smoothing out the space-time geometry. The consequences of such terms on the dynamics of the Universe are discussed.

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Our universe is manifestly spatially isotropic when observed over large scales ($L > 10$ Mpc). However, locally (L up to 10 Mpc and most likely up to 100 Mpc, as recent analyses of supercluster data have shown) it appears quite inhomogeneous with vast empty regions punctuated with richly structured concentrations of matter and radiation.

On such local scales, the space-time geometry, as determined by Einstein's equations, is extremely complex, and its structure is not, to a large extent, particularly illuminating for the purposes of cosmologists. Thus, it is customary to ignore this fine graining when dealing with the kinematics and the dynamics of the universe as a whole. In practice, we imagine averaging out all the inhomogeneities associated with the local material content of the actual universe, and redistributing them homogeneously (e.g., in the form of a uniform incoherent dust or perfect fluid). The basic and tacit assumption underlying this process is that such a smoothed out universe and the actual, locally inhomogeneous universe behave identically under their own gravitation. Existing astronomical data do not contradict such an assumption, so that it is usually taken for granted.

In this connection, it has been remarked (most recently by Ellis[1]) that nobody has ever provided an explicit constructive procedure for carrying out such a smoothing process in the full theory (there have been attempts in the linearized theory; see Ref. 1 for references), and that consequently, some care must be taken in reaching the above conclusions. The basic argument advocated in such criticisms is that Einstein's equations are highly nonlinear, and it is not clear what is the effect on them of averaging out either the external sources or the space-time geometry. For instance, Ellis[1] conjectured that on smoothing out the space-time geometry there would appear geometric correction terms in the sources to Einstein's equations. Such correction terms may influence the dynamics of the universe, for instance by altering the energy conditions satisfied by the physical sources, and should be properly taken into account.

In order to provide an answer to the questions raised by the above remarks, we discuss here, in the full theory, a smoothing-out procedure for a physically significant class of space-times. These space-times are associated with gravitational configurations that may be considered near to the standard gravitational configurations generating closed Friedmann-Robertson-Walker (FRW) universes. In order to define explicitly the smoothing out mapping we use techniques from the Arnowitt-Deser-Misner formulation of the initial-value problem in general relativity.[2] The basic idea is to pick up a suitable initial data set, the Cauchy development of which is the space-time to be averaged out. Such data set is then smoothly deformed, by means of parabolic-type operators, into a FRW initial-data set. This deformation is constructed in such a way as to make the deformed data satisfy the four constraints associated with Einstein's equations. It then follows, by standard theorems[2] on the Cauchy development of regular initial-data sets, that the flow of deformed data generates a one-parameter family of solutions of the field equations. Such a family of solutions interpolates between the original space-time and a closed FRW space-time that can be considered the smoothed out counterpart of the given model universe. In this Letter, we describe the above procedure, providing explicitly the geometric correction terms to the physical sources induced on averaging. We also prove that such correction terms do not give rise to a violation of the dominant energy condition.[3] This result supports the naturality of this smoothing out technique and it implies that the FRWs obtained on averaging are indeed physically admissible solutions of the field equations. Applications to particular examples such as anisotropic cosmological models and further details will be presented elsewhere.

Let $(V^4 \overset{\phi}{\simeq} M \times I,\ g)$ be a space-time manifold considered as the Cauchy development of a regular initial-data set (M, h, K). M is the three-manifold carrier of the initial data, ϕ is a diffeomorphism mapping V^4 onto the product $M \times I$ (I being a suitable subset of R), and h and K are symmetric bilinear forms on M representing in the final space-time the induced Riemannian three-metric on M, and the second fundamental form of the embedding of M in (V^4, g), respectively. Following the remarks in the introduction, we assume that M is topologically a three-sphere S^3 (possibly quotiented by a finite group of isometries, Γ, acting without fixed points), and that such manifold supports a particular class of initial data. Namely, we consider those data on M for which $\mathrm{Ric}(h)$ is a positive-definite bilinear form [$\mathrm{Ric}(h)$ is the Ricci tensor associated with h]. As a consequence of a result of Hamilton,[4] such data belong to the same connected component, in the space of all possible initial-data sets on M, as the standard FRW data on S^3. As such, a (V^4, g) resulting from the time evolution of any data set in that class may be considered as modeling a locally anisotropic and inhomogeneous universe not too far from a standard closed FRW space-time. Notice that such class is quite large, since it contains solutions of the field equations which are not simply perturbations of closed FRW space-times. For instance, the Taub universe belongs to it as well as other empty Bianchi IX models.

Under the above assumptions, it is possible to provide explicitly a smoothing out mapping that associates with the given initial-data set a one-parameter family of initial data $(M, h(\beta), K(\beta))$, with $0 \leq \beta < \infty$, $h(0) = h$, $K(0) = K$, approximating more and more closely, in the uniform topology, the standard initial-data set for a closed FRW model and reaching it uniformly as $\beta \to \infty$.

The main technical tool we need is the result by Hamilton,[4] already recalled (see also DeTurck[5]), showing that if $\mathrm{Ric}(h) > 0$ on a closed three-manifold M, then the metric in question can be continuously deformed into the standard constant-curvature metric $\bar{h}$ on S^3. This deformation is attained by the flow of metrics $h(\beta)$, $0 \leq \beta < \infty$, solution of the nonlinear, weakly parabolic, initial-value problem

$$\frac{\partial}{\partial \beta} h_{ab}(\beta) = \frac{2}{3} \langle R(\beta) \rangle_\beta h_{ab}(\beta) - 2R_{ab}(\beta),$$
$$h_{ab}(0) = h_{ab} \quad (a,b = 1,2,3), \tag{1}$$

where $\langle R(\beta) \rangle_\beta$ is the average scalar curvature over $(M, h(\beta))$, i.e., $\langle R(\beta) \rangle_\beta = \mathrm{Vol}^{-1}(M, h(\beta)) \times \int_M R(\beta) dv_\beta$, [henceforth, the angular brackets $\langle\ \rangle_\beta$ will always denote the average of the enclosed quantity over $(M, h(\beta))$], and where $R_{ab}(\beta)$ and $R(\beta)$ denote the components of the Ricci tensor and the scale curvature associated with $h(\beta)$, respectively. The smooth flow of metrics $h(\beta)$ defined on M by (1) (the Hamilton flow) has a number of remarkable properties. It can be shown[6] that as $\beta \to \infty$, $h(\beta)$ approaches $\bar{h}$ uniformly. Moreover, $h(\beta)$ preserves the total volume of (M,h), namely $\mathrm{Vol}(M, h(\beta)) = \mathrm{Vol}(M,h)$, $0 \leq \beta < \infty$. Also, all the symmetries which the original metric h may be endowed with are preserved by $h(\beta)$. Thus, the Hamilton flow appears as a natural candidate for a smoothing out mapping that associates with the given h a constant-curvature metric on S^3. Such metric is fixed by the natural normalization condition $\mathrm{Vol}(S^3, \bar{h}) = \mathrm{Vol}(M,h)$, and obtained from the original metric h by isotropizing its asymmetries while preserving all its preexisting isometries, if any.

In order to smooth out the complete set of initial data (M,h,K), we need also a way of averaging out the second fundamental form K. To this end, together with the flow $h(\beta)$ solution of (1), consider a nearby flow $h'(\beta)$ with initial condition $h'(0) = h + \epsilon K$, where K is the given value of the second fundamental form. These flows evolve with β yielding as "connecting vector" the bilinear form $K(\beta) = \lim_{\epsilon \to 0} [h'(\beta) - h(\beta)]\epsilon^{-1}$. The β evolution of $K(\beta)$ is found by linearizing (1) to obtain the initial-value problem

$$\frac{\partial}{\partial \beta} K_{ab}(\beta) = \tfrac{2}{3} h_{ab}(\beta) \left(\tfrac{1}{2} \langle R(\beta) k(\beta) \rangle_\beta - \tfrac{1}{2} \langle R(\beta) \rangle_\beta \langle k(\beta) \rangle_\beta - \langle R_{ab}(\beta) K^{ab}(\beta) \rangle_\beta \right.$$
$$\left. + \tfrac{2}{3} \langle R(\beta) \rangle_\beta K_{ab}(\beta) - \Delta_\beta K_{ab}(\beta) - L_Y h_{ab}(\beta), \quad K_{ab}(0) = K_{ab}, \right. \tag{2}$$

where $k(\beta) \equiv K^a{}_a(\beta)$, $Y_a \equiv \nabla_r K^r{}_a(\beta) - \frac{1}{2} \nabla_a k(\beta)$ [∇ is the Riemannian connection associated with $h(\beta)$], Δ_β denotes the De Rham–Lichnerowicz Laplacian associated with $h(\beta)$,[7] and L_Y is the Lie derivative along the vector field Y.

A straightforward calculation shows that the flow $K(\beta)$, solution of (2), is such that $(\partial/\partial\beta)[\langle k(\beta) \rangle_\beta] = 0$, namely the space average of its trace remains constant during the deformation.

Furthermore, since (2) is nothing but the formal linearization of Eq. (1) defining the Hamilton flow, it immediately follows from the isotropizing properties of this latter that $\lim_{\beta\to\infty} K_{ab}(\beta)=\frac{1}{3}(\langle k\rangle_0 \times \bar{h}_{ab})$. Thus, the flow $K(\beta)$, formally defined by (2), is such as to deform the given K by gradually eliminating its shear $\tilde{K}_{ab}=K_{ab}-\frac{1}{3}kh_{ab}$, and by replacing the original (position-dependent) rate of volume expansion k with its average value.

In order that the smoothing-out flow of data $(h(\beta), K(\beta))$ is such as to give rise to a curve $\beta\to(h(\beta), K(\beta))$ of regular initial-data sets, they must satisfy, for each value of β, the four constraints associated with Einstein's equations, namely

$$R(\beta)-K_{ab}(\beta)K^{ab}(\beta)+k^2(\beta)=2\rho(\beta), \qquad (3)$$

$$\nabla_a K^{ab}(\beta)-\nabla^b k(\beta)=J^b(\beta), \qquad (4)$$

where $\rho(\beta)$ and $J(\beta)$, respectively, are the mass density and the momentum density [as referred to the β-dependent measure associated with $h(\beta)$] of the sources as experienced by a system of ∞^3 observers instantaneously at rest on M. Notice that $\rho(0)=\rho$ and $J(0)=J$ are the actual densities corresponding to the physical sources describing the given gravitational configuration (M,h,K). Hence, for $\beta=0$, (3) and (4) are assumed to hold true. Notice also that once h and K are given, the left-hand members of (3) and (4) are, via the Hamilton flows (1) and (2), known functions of β. As a consequence we cannot define independently the averaging flows $\rho(\beta)$ and $J(\beta)$, since in that case (3) and (4) could not hold true, in general. This circumstance is linked with the fact that there is no simple way of providing *ad hoc* averaging procedures for the sources without explicitly taking into account the back reaction of the geometry. Since this back reaction is expressed by the constraints, the above remarks suggest that we interpret (3) and (4) as actually defining $\rho(\beta)$ and $J(\beta)$. The convenience of such interpretation follows immediately by verification that the flows $\rho(\beta)$ and $J(\beta)$ so defined give rise to homogenization and isotropization of the original ρ and J. For, from (3) and (4), and the properties of the Hamilton flows, it follows that as $\beta\to\infty$, $\rho(\beta)\to\bar{\rho}=\lim_{\beta\to\infty}\langle\rho(\beta)\rangle_\beta$, and $J(\beta)\to 0$, uniformly.

The main question with such a smoothing-out procedure for ρ concerns the relation between $\rho(\beta)$, as $\beta\to\infty$, and $\langle\rho\rangle_0$, the average value of the original physical ρ. It can be easily shown, by directly taking into account the properties of the flows (1), (2), that such a relation is provided by

$$\bar{\rho}=\lim_{\beta\to\infty}\langle\rho(\beta)\rangle_\beta=[\langle\rho\rangle_0+\tfrac{1}{2}\langle\tilde{K}_{ab}\tilde{K}^{ab}\rangle_0+\tfrac{1}{2}\bar{R}(\eta+\sigma^2)](1+\sigma^2)^{-1}, \qquad (5)$$

where $\sigma=(\langle k^2\rangle_0-\langle k\rangle_0^2)^{1/2}/\langle k\rangle_0$ is the standard deviation describing the fluctuations of the original value of the rate of volume expansion with respect to its (conserved) average value $\langle k\rangle_0$, and where $\eta=(\bar{R}-\langle R\rangle_0)/\bar{R}$ $(0\leq\eta<1)$ denotes the relative function of the physical scalar curvature with respect to the final averaged curvature $\bar{R}$.

Before turning to a critical discussion of the physical meaning of such a result, we must complete the averaging procedure for the external sources by providing a way of smoothing out the spatial stress tensor S corresponding to such sources. As is known, the stress tensor enters Einstein's equations only in their evolutive part, namely [if (M,h_t) defines a normal geodesic slicing of (V^4,g) for t sufficiently small],

$$(\partial/\partial t)K_{ab}=R_{ab}+kK_{ab}-2K_{ac}K_b^c-(S_{ab}-\tfrac{1}{2}sh_{ab})-\tfrac{1}{2}\rho h_{ab}, \qquad (6)$$

where $s=S_a^a$. We can define an isotropizing flow $S(\beta)$ for the stress tensor S_{ab} by requiring that, for each t for which the evolution of the curve of data $\beta\to(h(\beta),K(\beta))$ is defined, the flows $(h_t(\beta), K_t(\beta))$ [resulting from evolution equations similar to (6)] are Hamilton flows characterized by the initial-value problems (1), (2) with initial conditions $h_t(0)=h_t$, $K_t(0)=K_t$, respectively.

It is not difficult to understand how this procedure works. Let $\mathrm{Vol}(M,h_t)$ denote the volume of (M,h_t) as t varies. If, as required, $(h_t(\beta),K_t(\beta))$ remain Hamilton flows under the time evolution of the data $(h(\beta), K(\beta))$, then, by the properties of such flows, $\mathrm{Vol}(M,h_t(\beta))=\mathrm{Vol}(M,h_t)$, for each t for which the evolution is defined, and $0\leq\beta<\infty$. Letting $\beta\to\infty$, we can determine in this way the time dependence of the volume of the surfaces of homogeneity of the closed FRW model generated by smoothing out the data (M,h,K). In this way, the dynamics of the closed FRW associated, via averaging, to (M,h,K) is tied to the dynamics of the original space-time by the natural condition $\mathrm{Vol}(S^3,\bar{h}_t)=\mathrm{Vol}(M,h_t)$. Notice that this last relation provides a precise mathematical content to the requirement that the original physical model universe and its FRW smoothed-out counterpart should behave as similarly as possible under their own gravitation. From $\mathrm{Vol}(M,h_t)=\mathrm{Vol}(M, h_t(\beta))$, we also get that $(\partial/\partial t)(\langle k\rangle_0)=(\partial/\partial t) \times(\langle k(\beta)\rangle_\beta)$ along the flow $(h_t(\beta),K_t(\beta))$. This last relation easily provides the smoothed-out pressure $\bar{p}$ associated[8] with the sources corresponding to the particular FRW obtained. For, from it, by taking into account (6) and letting $\beta\rightarrow\infty$, we get

$$\bar{p}=\tfrac{1}{3}\langle s\rangle_0+\tfrac{2}{3}\langle\tilde{K}_{ab}\tilde{K}^{ab}\rangle_0 -\tfrac{4}{9}\sigma^2\langle k\rangle_0^2+\tfrac{1}{3}(\langle\rho\rangle_0-\bar{\rho}), \qquad (7)$$

where $\bar{\rho}$ is given by (5).

It follows from (5) and (7) that $\bar{p}>0$ and that the dominant energy condition $|\bar{p}|\leq\bar{\rho}$ is satisfied, so that the smoothed out sources are, in this sense, physically admissible. More particularly, if, with our real universe in mind, we assume that $\sigma^2\approx 0$ (i.e., homogeneity of the expansion factor) and that $\eta<<1$ (i.e., small fluctuations, on the average, of the physical curvature with respect to the Friedmannian background curvature), then (5) and (7) yield $\bar{p}\approx\frac{1}{3}\langle s\rangle_0+\frac{1}{2}\langle\tilde{K}_{ab}\tilde{K}^{ab}\rangle_0$, and $\bar{\rho}\approx\langle\rho\rangle_0+\frac{1}{2}\langle\tilde{K}_{ab}\tilde{K}^{ab}\rangle_0$. Thus if our universe were closed, it could be modeled correctly by a closed FRW only if we add to the physical sources $(\langle\rho\rangle_0, \langle s\rangle_0)$ the term $\langle\tilde{K}_{ab}\tilde{K}^{ab}\rangle_0$ that takes into account the contribution of cosmological graviational radiation.[9] Notice that there is no experimental evidence,[10] up to now, showing that $(\langle\tilde{K}_{ab}\tilde{K}^{ab}\rangle_0/\langle\rho\rangle_0)<<1$, and hence such a term can influence quite seriously the dynamics of the Universe.

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[1]G. F. R. Ellis, in *Proceedings of the Tenth International Conference on General Relativity and Gravitation,* edited by B. Bertotti, F. De Felice, and A. Pascolini (Reidel, Dordrecht, 1984). See also H. Sato, *ibid.*

[2]Y. Choquet-Bruhat and J. W. York, Jr., in *General Relativity and Gravitation,* edited by A. Held (Plenum, New York, 1980), p. 99.

[3]S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge Press, Cambridge, 1973), p. 91.

[4]R. S. Hamilton, J. Diff. Geom. **17**, 255 (1982). Notice that Hamilton's theorem actually forces M to be topologically S^3/Γ. Notice also that there are not particular restrictions on the allowed matter content of the spacetime manifold associated with such data (the usual energy condition is assumed).

[5]D. DeTurck, J. Diff. Geom. **18**, 157 (1983).

[6]See Ref. 4, p. 304.

[7]See Ref. 2, p. 123.

[8]$\bar{p}$ denotes the value of the pressure, in the final FRW, on the surface of homogeneity $t=0$. $\bar{p}_t$, as well as $\bar{\rho}_t$, can be determined by means of the evolution equation once the state equation for the physical sources is provided.

[9]Since experimental evidence suggests that our universe may be open, it would be interesting to extend the above results to this last case. We may conjecture that this can be done by showing the existence of a suitable Hamilton flow for the typical inhomogeneity region.

[10]B. Bertotti and B. J. Carr, Astrophys. J. **236**, 1000 (1980).