Analytic Approach to Phase Transitions in Lattice Gauge Theories

R. Gambini and A. Trias
Departamento de Física, Universidad Simón Bolívar, Caracas 108, Venezuela
(Received 26 June 1984)

The Hamiltonian formulation is analyzed in a basis of gauge-invariant loop states. A natural geometric asymptotic approximation for the eigenstates of the Hamiltonian is developed to identify the phase diagram of the theory. Satisfactory explicit calculations are performed in the case of Z_2.

PACS numbers: 11.15.Ha, 05.70.Fh, 11.15.Tk

Considerable progress is currently being achieved in lattice gauge theories. One now has numerical evidence for confinement in SU(2) and SU(3) gauge theories,[1,2] chiral symmetry breaking in QCD,[3-5] and even valuable estimates of the meson and baryon mass spectra.[3,4,6,7] Unfortunately these encouraging numerical achievements have not always been matched by a corresponding understanding of the special features of lattice gauge theories at the analytical level in spite of the progress achieved in strong-coupling[8,9] and weak-coupling[10,11] expansions, mean-field[12] approximations, and variational methods.[13,14]

In this paper a simple analytical method is proposed to deal with phase transitions in lattice gauge theories. Although it may be generalized to deal with any Z_N system and hopefully also with continuous Abelian and non-Abelian theories, only the Z_2 case will be considered. In spite of its simplicity, the method reproduces the main results concerning the phase diagram of the Z_2 system either known exactly by rigorous arguments or suggested by Monte Carlo calculations.[15,16]

Quantum gauge theories are formulated in the "large" Hilbert space of all field configurations. This is the usual procedure for both continuous and lattice gauge theories. Physical states are then associated with equivalence classes of gauge-transformed states. For some problems a more direct characterization of physical states may be advantageous. Recently[17] the present authors have proposed to work in a basis of gauge-invariant states $|C\rangle$ labeled by closed contours regarded as elements of the group of loops.[18] Let us refer from now on to this choice as the C representation. The Hamiltonian formulation in continuous space for a general non-Abelian gauge theory has been worked out in the C representation, and it proved useful in the discussion of several nonperturbative questions including the confinement problem.[17] This formulation may be specialized to lattice gauge theories. In fact the C representation is even more natural in the lattice since all concepts and techniques associated with the group of loops may be rigorously defined in discrete space.

The Z_2 gauge theory provides a most elementary illustration of the Hamiltonian formulation in the C representation. For this system the "large" Hilbert space is spanned by a basis of states

$$|n\rangle = |n_1, n_2, \ldots, n_\Lambda\rangle, \tag{1}$$

where n_λ is a binary digit associated with link λ and Λ is the total number of links in the lattice. Gauge transformations are realized in terms of operators that flip all spins attached to a given vertex. It may be proven that the physical space is spanned by a dual basis of states $|C\rangle$ which may be defined by

$$\langle n|C\rangle = 2^{-\Lambda/2} \prod_{\lambda \in C} e^{i\pi n_\lambda}. \tag{2}$$

The closed paths C are elements of the "Z_2 version" of the group of loops, an Abelian subgroup of the group of loops where all elements verify $CC = 1$. Hence the two orientations on C are equivalent and a given link may appear only once in a loop.

The relevant operators of the Hamiltonian formulation in the C representation are Wilson's loop operator,

$$W(C)|C'\rangle = |CC'\rangle, \tag{3}$$

and its temporal loop derivative which for any Abelian theory may be traded by the electric field operator given, in the Z_2 case, by

$$E(\lambda) = [1 - a(\lambda)]/2, \tag{4}$$

where $a(\lambda)$ is the operator that flips the spin at link λ. The C states are eigenstates of the electric field operator:

$$E(\lambda)|C\rangle = N_\lambda|C\rangle, \tag{5}$$

where N_λ is 1 if λ belongs to C and 0 otherwise. The Hamiltonian associated with the gauge theory is given by

$$H = -\beta \sum_p W(p) + \sum_\lambda E(\lambda), \tag{6}$$

where the magnetic part is given by a sum of Wilson operators over all plaquettes and the electric part contains a sum over all links of the lattice of the electric field operator. Equation (6) coincides with Wegner's Hamiltonian[19] up to irrelevant constants.

According to (3) and (5), the Schrödinger equation in the C representation may be written as

$$-\beta\sum_p \Psi(p,C)+L\Psi(C)=E\Psi(C), \tag{7}$$

where $\Psi(C)=\langle C|\Psi\rangle$ and L is the length of C.

According to Wegner[19] the phase diagram is dominated by the asymptotic behavior of Wilson's loop average which in the C representation is obtained as a self-convolution of the ground state. Hence we are interested in solving this equation for large loops. To be more precise, let us define the asymptotic region in the loop space as consisting of those large loops that may be obtained by dilatation of finite loops. Our purpose is to set the appropriate asymptotic form of (7) in this region. In general the eigenstates $\Psi(C)$ must depend on a complete set of parameters providing a unique characterization of the loop. The list may include L the length, A the minimal number of plaquettes with net border in C, the number of corners, the size and relative position of "bottlenecks" (parallel lines in C separated by one unit of the lattice), etc. However, as we shall see, one may argue that only L and A are relevant in the asymptotic region as defined above. In this case the appropriate asymptotic form of (7) for a lattice of volume V in d spatial dimensions is

$$\begin{aligned}-M\Psi(L+4,A+1)&-2(d-1)L[\Psi(L+2,A+1)-\Psi(L+4,A+1)]\\&-L[\Psi(L+2,A-1)-\Psi(L+2,A+1)]\\&-(A-L)[\Psi(L+4,A-1)-\Psi(L+4,A+1)]+\mu L\Psi(L,A)=\mu E\Psi(L,A),\end{aligned} \tag{8}$$

where $M=Vd(d-1)/2$ is the total number of plaquettes and $\mu=1/\beta$ has been introduced for convenience.

The four first terms correspond to the action of the magnetic part of the Hamiltonian. The first one considers all plaquettes of the lattice as lying outside the minimal area of C and without contact with its border. The second term corrects the first one by taking into account the plaquettes hitting the border of the loop. For each link in C there are $2(d-1)$ such plaquettes and these increase L only by two units. The third term corrects the second one since one of these plaquettes will make contact with the minimal area of C erasing one unit of A. At last, the fourth term corrects the first one by considering the plaquettes hitting the minimal-area region of C without contact with its border. The count is incorrect for plaquettes hitting corners. These could be taken into account by including terms proportional to the number of corners. However, the number of corners remains fixed under dilatations. Therefore the relative contribution should be negligible in the asymptotic region as compared with the terms proportional to L and A. Bottlenecks and many other configurations of finite loops are also not counted in (8). However, these configurations disappear by dilatation of the loop and should not be incorporated in (8) which is assumed to hold in the asymptotic region as defined above.

Let us introduce

$$\epsilon=-\mu E/M, \tag{9}$$

which, according to (7), represents the average plaquette in the state $|\Psi\rangle$ in the limit $V\to\infty$. Equation (8) may be written in the form

$$\begin{aligned}2(d-1)L[\Psi(L+4,A+1)-\Psi(L+2,A+1)]&+L[\Psi(L+2,A+1)-\Psi(L+2,A-1)]\\&+(A-L)[\Psi(L+4,A+1)-\Psi(L+4,A-1)]+\mu L\Psi(L,A)\\&=M[\Psi(L+4,A+1)-\epsilon\Psi(L,A)].\end{aligned} \tag{10}$$

A possible method to deal with this partial finite-difference equation may be to introduce the Mellin transform

$$\Psi(L,A)=\int\int d\xi\, d\eta\, G(\xi,\eta)\xi^L\eta^A \tag{11}$$

to obtain a complete family of solutions of the corresponding partial differential equation for $G(\xi,\eta)$, and use standard asymptotic methods to obtain the behavior in the limit $M\to\infty$. This method leads to the one-

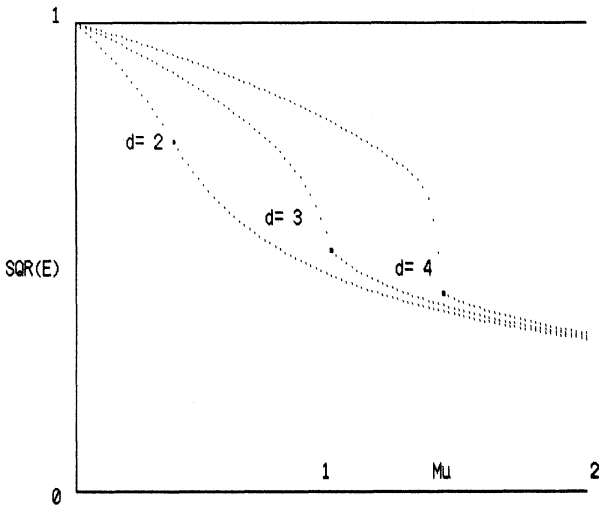

FIG. 1. $\sqrt{\epsilon}$ vs μ. Square dots are the transition points.

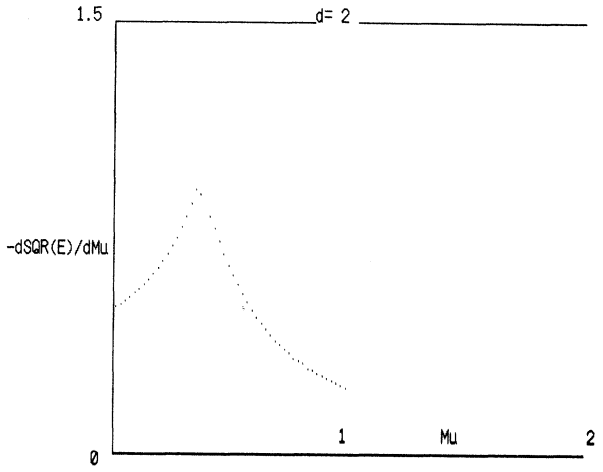

FIG. 2. First derivative of the average plaquette vs μ.

parameter family of asymptotic solutions

$$\Psi(L,A)=x^{L/2}y^{A}, \tag{12}$$

where $0\leq y\leq 1$ since we are looking for normalizable states. x is defined in terms of y by

$$\frac{dx}{dy}=\frac{2x\{(1-x)[1+(2d-3)y^2]-\mu y\}}{y(1-y^2)}, \tag{13}$$

$$x(0)=1,$$

and ϵ is given by

$$\epsilon(y)=x^2(y)y. \tag{14}$$

It should be pointed out that a modification of Eq. (13) by a global factor $x^{\alpha}y^{\beta}$ on the right-hand side should in principle be equally acceptable. In fact this factor may be obtained by the changing of L,A to $L-L_0,A-A_0$ in Eq. (8). These shifts may even be produced independently in the terms proportional to L and A and should be undetectable since (8) is assumed to hold in the asymptotic region. Fortunately this expected asymptotic ambiguity may be removed in a unique way. For $\alpha\neq 1$ the solutions of (13) do not exist for all y in (0,1) in the strong-coupling region and the family does not cover long-range excitations of the ground state which physically must always be present. For $\beta\neq 1$ the solutions are not analytic at $y=0$. Hence smoothness and full definition of the obtained family lead us uniquely to (13).

For the ground state ϵ is proportional to the free energy and therefore may be used to label the phases of the system. The best approximation for the ground state is obtained, according to (9), when one chooses the value of y that maximizes the average plaquette. This furnishes ϵ as a function of μ and the analysis of the phase diagram may be performed.

Following this procedure one obtains for $d=3$ a two-phase system with a first-order transition located at $\mu=1$ which is known to be the exact value. For $d>3$ the transition point is located at

$$\mu_{\rm tr}=\tfrac{1}{2}(d-1) - \{d+1+[(d+3)(d-1)]^{1/2}\}^{-1}, \tag{15}$$

which behaves as $\frac{1}{2}(d-1)$ as d increases. The discontinuity in the first derivative of the free energy becomes more abrupt as d grows. These results are depicted in Fig. 1. For $d=2$ one obtains a rather different picture. As shown in Figs. 1 and 2, the system undergoes a second-order transition at $\mu=(\sqrt{5}-1)/4$. The optimal value of y departs smoothly from $y=1$ at the transition point [Fig. 3(a)]. This should be compared with the abrupt change for $d\geq 3$ [Fig. 3(b)].

In summary one has obtained a satisfactory description of the phase diagram of the Z_2 lattice gauge theory for all d. Even an indication of a roughening transition may be observed, deep in the strong-coupling region, due to the change from ∞ to 0 in the slope of $\epsilon(y)$ at $y=1$ [Fig. 3(a) and 3(b)], which may induce nonanalytical behavior in this region. Vacuum expectation values such as monopole charge density, string tension, etc., may be considered within this method from (12) and the exploitation of the geometric ideas just developed. Preliminary steps in this direction seem satisfactory.

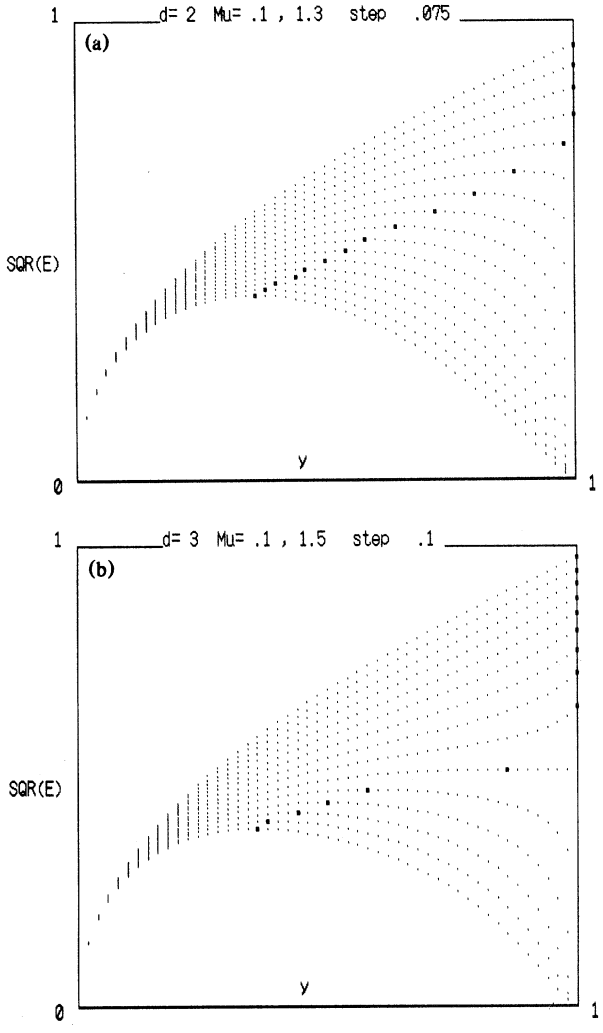

FIG. 3. $\sqrt{\epsilon}$ vs y. Square dots are the optimal values of y. (a) $d=2$; (b) $d=3$.

It is a pleasure to thank Dr. A. González-Arroyo for communicating to us part of his knowledge and all of his enthusiasm for the lattice.

[1]M. Creutz, Phys. Rev. D **21**, 2308 (1980).
[2]M. Creutz, Phys. Rev. Lett. **45**, 313 (1980).
[3]H. Hamber and G. Parisi, Phys. Rev. Lett. **47**, 1792 (1981).
[4]E. Marinari, G. Parisi, and C. Rebbi, Phys. Rev. Lett. **47**, 1795 (1981).
[5]J. Kogut, M. Stone, H. W. Wyld, J. Shigemitsu, S. H. Shenker, and D. K. Sinclair, Phys. Rev. Lett. **48**, 1140 (1982).
[6]D. Weingarten and D. Petcher, Phys. Lett. **99B**, 33 (1981).
[7]D. Weingarten, Phys. Lett. **109B**, 57 (1982).
[8]M. Creutz, Rev. Mod. Phys. **50**, 561 (1978).
[9]G. Munster, Nucl. Phys. **B190** [FS3], 439 (1981), and **B200** [FS4], 536(E) (1982), and **B205** [FS5], 648(E) (1982).
[10]B. Baaquie, Phys. Rev. D **16**, 2612 (1977).
[11]A. González-Arroyo and C. P. Korthals Altes, Nucl. Phys. **B205**, 46 (1982).
[12]J. M. Drouffe and J. B. Zuber, Phys. Rev. **102**, 1 (1983).
[13]S. Drell, H. Quinn, B. Svetitsky, and M. Weinstein, Phys. Rev. D **19**, 619 (1979).
[14]V. Heller, Phys. Rev. D **23**, 2357 (1981).
[15]M. Creutz, L. Jacobs, and C. Rebbi, Phys. Rev. Lett. **42**, 1390 (1979).
[16]M. Creutz, L. Jacobs, and C. Rebbi, Phys. Rev. D **20**, 1915 (1979).
[17]R. Gambini and A. Trias, Phys. Lett. **141B**, 403 (1984).
[18]R. Gambini and A. Trias, Phys. Rev. D **23**, 553 (1981).
[19]F. Wegner, J. Math. Phys. (N.Y.) **12**, 2259 (1971).