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Path-Integral Derivation of the Dirac Propagator

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The Dirac propagator $(\gamma p - m)^{-1}$ has been calculated as a path integral over a recently proposed classical action. Thus a new formulation is given to the longstanding problem of extending the path integration to discrete quantum spin as an integral over a continuous variable.

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There is, to our knowledge, no derivation of the propagator of a Dirac particle, $(\gamma p - m)^{-1}$, as a path integral over a classical action S in the form $\int \exp(iS)$, although a number of authors have made interesting and ingenious stochastic checkerboard-type calculations to account for the spin degree of freedom. Feynman and Hibbs[1] consider in the case of $1+1$ dimensions, all particle paths going in discrete zigzags to the right or to the left with velocity of light.[2] This picture has recently been generalized to $3+1$ dimensions.[3] The corresponding stochastic process reached by analytic continuation has been given recently.[4] Related methods have been given by other authors.[5–7] There is an element of *Zitterbewegung* in all these works. Still others have started from quantum theory and, going backwards, tried to formulate the quantum propagator as a path integral over some effective action.[8–11] Grassmann numbers have also been used to formulate path integrals for a free particle.[12] No classical action S was used in any of these calculations. In fact Feynman and Hibbs state that "for a relativistic particle with spin (described by the Dirac equation), the amplitude cannot be described by a simple path integral based on any reasonable action" (Ref. 1, p. 264). Further, ". . .and because spin takes discrete values it has been difficult to suggest for it a continuous path subsequently to be summed over so as to obtain the propagator" (Ref. 4, p. 182).

Since now a classical action has become available[13] which, by canonical quantization, gives the exact Dirac equation, we show is this work that this theory also gives in a very simple way the Dirac propagator. Thus we show in a different way the correctness of the underlying classical picture for the Dirac equation, and we provide a solution, within this theory, to this longstanding spin problem in the theory of path integration.

The classical Lagrangian is given by

$$L = -\frac{\lambda}{2i}(\dot{\bar{z}}z - \bar{z}\dot{z}) + p_\mu(\dot{x}^\mu - \bar{z}\gamma^\mu z) e\bar{z}\gamma^\mu z A_\mu. \quad (1)$$

Here λ is a constant of dimension of action $(c=1)$ and z is a classical four-component spinor variable with $\bar{z} = z^\dagger \gamma^0$ its conjugate. The velocity is represented by $\bar{z}\gamma^\mu z$ and the Lagrange multiplier p_μ turns out to be the momentum. The Lagrangian does not contain the mass m which reenters later as the value of the constant of the motion $\mathcal{H} = p^\mu \bar{z}\gamma_\mu z$. The classical motion contains a real *Zitterbewegung* of the charge around a "center of mass" whose orbital angular momentum is related to the spin. Finally the proper time used in Eq. (1) is that of the center of mass. The configuration space of the classical particle is $M_4 \otimes C_4$ and (x_μ, p_μ) and $(z, -i\bar{z})$ are conjugate pairs of variables.

The basic physical difference between the Dirac propagator and the nonrelativistic propagator lies in the fact that the former is a matrix meaning that we must specify not only the end points of the paths (x_a, x_b), but also the initial and final spin orientations of all paths (more generally all other internal degrees of freedom of the system).

We shall therefore consider path integrals in the space $M_4 \otimes C_4$ with end points $x_a, x_b \in M_4$, and $z_\alpha, \bar{z}_\beta \in C_4$,

$$K(x_b, x_a) = -(i/\lambda)\int_0^\infty d\tau\, K^\tau_{\alpha\beta}(x_b, x_a),$$

where

$$K^\tau_{\alpha\beta}(x_b, x_a) = \int \mathcal{D} z_\alpha \mathcal{D}\bar{z}_\beta \mathcal{D} x\, \mathcal{D} p \exp\left(i\int_0^\tau d\tau L_{\beta\alpha}\right). \tag{2}$$

Here τ is total transit time, the τ integration takes the place of a sum over the number of zigzags in Ref. 1 and we kept a $\mathcal{D} p$ integration for convenience, and

$$L_{\beta\alpha} = -(\lambda/2i)(\dot{\bar{z}}_\beta \delta_{\beta\alpha} z_\alpha - \bar{z}_\beta \delta_{\beta\alpha}\dot{z}_\alpha) + p_\mu(\dot{x}^\mu - \bar{z}_\beta \gamma^\mu_{\beta\alpha} z_\alpha), \tag{3}$$

so that the Lagrangian (1) is given by

$$L = \sum_{\alpha,\beta=1}^{4} L_{\beta\alpha}. \tag{4}$$

In (2) the $\mathcal{D} x$ and $\mathcal{D} p$ integrations can be performed immediately, in the usual way, by writing the x- and p-dependent factors of $K_{\alpha\beta}$ as

$$\int \prod_{j=1}^{n+1} \frac{d^4 p_j^\mu}{(2\pi)^4} \prod_{j=1}^{n+1} d^4 x_{\mu j} \exp\left[i\sum_{j=1}^{n+1}\{p_j^\mu (x_j - x_{j-1})_\mu - p_{\mu j}\bar{z}_\beta \gamma^\mu_{\beta\alpha} z_\alpha\}\right]$$
$$= \int \frac{d^4 p}{(2\pi)^4} \exp[ip^\mu (x_b - x_a)_\mu] \exp(-ip_\mu \bar{z}_\beta \gamma^\mu_{\beta\alpha} z_\alpha), \tag{5}$$

where

$$x_{n+1} = x_b, \quad x_0 = x_a.$$

Consequently we have

$$K^\tau_{\alpha\beta}(x_b - x_a) = \int d^4 p\, (2\pi)^{-4} K^\tau_{\alpha\beta}(\gamma\cdot p) \exp[ip_\mu (x_b - x_a)^\mu], \tag{6}$$

and the reduced path integral is

$$K^\tau_{\alpha\beta}(\gamma\cdot p) = \int \mathcal{D}\bar{z}_\beta \mathcal{D} z_\alpha \exp\left(i\int_{\tau_a}^{\tau_b} d\tau \tilde{L}_{\beta\alpha}\right), \tag{6'}$$

with

$$\tilde{L}_{\beta\alpha} = -(\lambda/2i)(\dot{\bar{z}}_\beta \delta_{\beta\alpha} z_\alpha - \bar{z}_\beta \delta_{\beta\alpha}\dot{z}_\alpha) - p_\mu \bar{z}_\beta \gamma^\mu_{\beta\alpha} z_\alpha. \tag{6''}$$

We write

$$\dot{\bar{z}} z - \bar{z}\dot{z} = d(\bar{z}z)/d\tau - 2\bar{z}\dot{z}$$

and incorporate the total time derivative into a measure on the internal space C_4 defined by

$$\int \mathcal{D}\bar{z}_\beta \mathcal{D} z_\alpha = \int_{-\infty}^{\infty} \prod_{j=1}^{n+1} \frac{i\lambda}{(2\pi)} d\bar{z}_{\beta j} dz_{\alpha j} \exp[-\tfrac{1}{2}\lambda(\bar{z}_\beta \delta_{\beta\alpha} z_\alpha)|_{\tau_a}^{\tau_b}] \exp(-i\tau m/\lambda). \tag{7}$$

The mass is introduced at this point by the measure $\exp(-i\tau m/\lambda)$, where λ is the parameter of dimension of an action in (1). With

$$\tau_b - \tau_a = \tau = (n+1)\epsilon,$$

the reduced path integral can now be defined by the limits

$$K^{\tau}_{\alpha\beta}(\gamma\cdot p)=\lim_{\substack{n\to\infty\\ \epsilon\to 0}}\int_{-\infty}^{\infty}\prod_{j=1}^{n+1}\left(\frac{i\lambda}{2\pi}d\bar{z}^j_\beta\,dz^j_\alpha\right)\exp\left(-i\tau\frac{m}{\lambda}\right)\prod_{j=1}^{n+1}\exp\left[i\left\{\frac{\lambda}{i}\bar{z}^j_\beta\,\delta_{\beta\alpha}(z^j_\alpha-z^{j-1}_\alpha)-\epsilon p_\mu\bar{z}^j_\beta\,\gamma^\mu_{\beta\alpha}z^j_\alpha\right\}\right]$$

$$=\lim_{\substack{n\to\infty\\ \epsilon\to 0}}\int_{-\infty}^{\infty}\prod\left(\frac{i\lambda}{2\pi}d\bar{z}^j_\beta dz^j_\alpha\right)\exp\left(-i\tau\frac{m}{\lambda}\right)\prod\exp[-i\bar{z}^j_\beta\{(i\lambda I+\epsilon p\cdot\gamma)_{\beta\alpha}z^j_\alpha-i\lambda\delta_{\beta\alpha}z^{j-1}_\alpha\}].\quad(8)$$

We integrate over $d\bar{z}_{\beta j}$:

$$K^{\tau}_{\alpha\beta}(p\cdot\gamma)=\exp\left(-i\tau\frac{m}{\lambda}\right)\lim_{\substack{n\to\infty\\ \epsilon\to 0}}\int\prod_{j=1}^{n+1}\frac{dz^j_\alpha}{[I+(\epsilon/i\lambda)(p\cdot\gamma)]_{\beta\alpha}}\,\delta[z^j_\alpha-i\lambda\delta_{\beta\alpha}\{(i\lambda I+\epsilon p\cdot\gamma)^{-1}\}_{\beta\alpha}z^{j-1}_\alpha].\quad(9)$$

Now the δ functions can be integrated, giving

$$K^{\tau}_{\alpha\beta}(p\cdot\gamma)=\lim_{\substack{n\to\infty\\ \epsilon\to 0}}\prod_{1}^{n+1}\frac{1}{I+(\epsilon/i\lambda)p\cdot\gamma}\exp\left(-i\tau\frac{m}{\lambda}\right).\quad(10)$$

But the limit in Eq. (10) is just the exponential function

$$\lim_{n\to\infty}[I+(\epsilon/i\lambda)p\cdot\gamma]^{n+1}=\exp[(\tau/i\lambda)p\cdot\gamma].\quad(11)$$

Thus

$$K^{\tau}_{\alpha\beta}(p\cdot\gamma)=\left[\exp\left(i\frac{\tau}{\lambda}p\cdot\gamma\right)\right]_{\alpha\beta}\exp\left(-i\tau\frac{m}{\lambda}\right)=\left[\exp\left(i\frac{\tau}{\lambda}[p\cdot\gamma-m]\right)\right]_{\alpha\beta},\quad(12)$$

which we insert in Eq. (6),

$$K^{\tau}_{\alpha\beta}(x_b-x_a)=\int dp\,(2\pi)^{-4}\exp[ip_\mu(x_b-x_a)^\mu]\{\exp[(i\tau/\lambda)(\not{p}-m)]\}_{\alpha\beta}\quad(13)$$

($\not{p}=\gamma^\mu p_\mu=\gamma\cdot p$), or, in matrix notation,

$$K^{\tau}(x_b-x_a)=\int dp\,(2\pi)^{-4}\exp[ip\cdot(x_b-x_a)]\exp[(i\tau/\lambda)(\not{p}-m)].\quad(13')$$

Finally we integrate over all possible τ's to obtain the final result,

$$K(x_b-x_a)=-\frac{i}{\lambda}\int_0^\infty d\tau K^\tau(x_b-x_a)=-\frac{i}{\lambda}\int_0^\infty d\tau\int\frac{dp}{(2\pi)^4}\exp[ip\cdot(x_b-x_a)]\exp\left[i\frac{\tau}{\lambda}(\not{p}-m)\right]$$

$$=\int\frac{dp}{(2\pi)^4}\exp[ip\cdot(x_b-x_a)]\frac{1}{\not{p}-m},\quad(14)$$

which is indeed the Dirac propagator for the electron. Our method also works for the electromagnetic coupling of the electron, and the Green's function expansion of QED can be derived from the classical action (1). This will be reported elsewhere.

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