Bifurcation Superstructure in a Model of Acoustic Turbulence

Werner Lauterborn and Engelbert Suchla
Drittes Physikalisches Institut, Universität Göttingen, D-3400 Göttingen, Federal Republic of Germany
(Received 27 June 1984)

A model of acoustic turbulence is investigated for its bifurcation structure by the calculation of spectral as well as ordinary bifurcation diagrams and (subharmonic) attractor maps. A superstructure resulting from nonlinear resonances is found with period-doubling Feigenbaum direct cascades and Grossmann inverse cascades as fine structure. Connected with the superstructure is a new family of periodic chaos with a different type of chaos belonging to each basic subharmonic period of oscillation.

PACS numbers: 47.55.Bx, 43.25.+y, 47.25.Mr

When a liquid is irradiated with sound of high intensity it may rupture to form bubbles or cavities. The phenomenon is called acoustic cavitation[1] and is accompanied by intense noise emission, the acoustic cavitation noise. The noise seems not to be due to a statistical rupture process in the liquid but to be of deterministic origin.[2] The evidence stems from the subharmonic route to chaos (broadband acoustic noise) which is observed when the sound pressure amplitude (taken as the control parameter) is raised. Thus an example of acoustic turbulence had been found experimentally.[2]

The question left is how to describe the experiment theoretically, in order to get a deeper understanding of the nature of the phenomenon. This paper investigates a theoretical model which, although relatively simple, displays diverse routes to and through chaos and complicated bifurcation structures. As is made evident below these may be explained as resulting from a superstructure of nonlinear hysteretic resonances with Feigenbaum[3] direct and Grossmann[4] inverse cascades as fine structure. In the chaotic regions the solutions of the model can also be transformed via subharmonic Poincaré maps into one-dimensional quadratic-looking maps. This connection is considered the main proof that the model belongs to the class of deterministically chaotic systems.[5]

The model is obtained by a set of simplifying assumptions to make it tractable. In the experiment, once the rupture process has started, thousands of tiny bubbles are generated, oscillating and moving in a complicated manner.[6] As a first approximation the mutual interaction of the bubbles is neglected. Then only the dynamics of a single bubble in a sound field needs to be considered. As a further approximation translational motions are neglected, and the bubble is taken as spherical. Even this system is far too complex to be written down easily as a result of heat and mass transfer across the boundary of the bubble. When these effects are neglected a reasonable model for an oscillating spherical bubble in a cold liquid can be formulated.[7] It is given by a highly nonlinear ordinary differential equation of second order for the radius R of the bubble as a function of time t:

$$R\left(1-\frac{U}{C}\right)\frac{d^2R}{dt^2}+\frac{3}{2}\left(1-\frac{U}{3C}\right)\left(\frac{dR}{dt}\right)^2-\left(1+\frac{U}{C}\right)H-\frac{U}{C}\left(1-\frac{U}{C}\right)R\frac{dH}{dR}=0,\quad \frac{dR}{dt}=U. \tag{1}$$

U is the bubble wall velocity, C is the sound velocity at the bubble wall, and H is the free enthalpy which for water is given by

$$H=\frac{n}{n-1}\frac{A^{1/n}}{\rho_0}\{[p(R)+B]^{(n-1)/n}-p_\infty+B)^{(n-1)/n}\}. \tag{2}$$

A, B, and n are constants ($A=3001$ bars, $B=3000$ bars, and $n=7$), and ρ_0 is the density at p_0 ($\rho_0=0.998$ g cm^{-3}, $p_0=1$ bar). The pressure $p(R)$ at the bubble wall is given by

$$p(R)=\left(p_0+\frac{2\sigma}{R}\right)\left(\frac{R_n}{R}\right)^{3\gamma}-\frac{2\sigma}{R}-\frac{4\mu U}{R}, \tag{3}$$

where p_0 is the static external pressure (1 bar), R_n the equilibrium radius of the bubble, σ the surface tension (72.5 dyn cm^{-1}), γ the ratio of the specific heats of the gas in the bubble (1.33), and μ the viscosity of the liquid (0.01 P). The pressure at infinity p_∞ is taken as $p_\infty=p_0-p_a\sin(2\pi f_a t)$, with p_a the acoustic pressure amplitude of the applied sound field and f_a its frequency. The sound velocity in the liquid at the bubble wall is given by

$$C=[C_0^2+(n-1)H]^{1/2}, \text{ with } C_0=1482 \text{ m/s}.$$

The above model has been investigated for its sound radiation and resonance behavior[8] in the spirit of earlier work[9] and the first results connecting it with the experiments[2] have been given.[10] To follow as closely as possible the experimental procedure the bubble, described by Eqs. (1) to (3), is subject to an increasing or decreasing sound pressure amplitude p_a. The numerically obtained radius-time curve is transformed into a series of short-time spectra as were the pressure-time data of the experiment[2] and plotted in the same way as the "theoretical visible noise." Only the bubble radius is considered here and not the sound radiated, but this does not affect the subharmonic bifurcation structure as the radiated sound is directly coupled to the radial motion. We suggest calling this kind of plot a *spectral bifurcation diagram.* It may be used with any chaotic system and not just noise[2] or bubble radii.

Figure 1 gives an example of a calculated spectral bifurcation diagram for the following parameters: bubble radius at rest $R_n=100$ μm; driving frequency $f_a=23.56$ kHz (a value used in experiments); and sound pressure amplitude p_a raised from 0 to 14.8 bars in 40 ms. Each single spectrum has been calculated from 2K data points comprising 0.68 ms. By shifting the window of 2K data by 1K in the data base of calculated radii a new spectrum is obtained and plotted every 0.34 ms. The resolution in the frequency domain is $\frac{1}{16}$ of the driving frequency. The individual spectra are normalized to the strongest line in the spectrum, in this case always the line at the normalized driving frequency 1. The gray scale unfortunately does not give a good representation in print of the relative intensities of the spectral lines (compare Ref. 2). Thus we are now working with color graphics. These, however, are not reproducible here. A quite complicated behavior is observed with a period-doubling bifurcation sequence to chaos, a window in the chaotic region, and then, starting at about 20 ms (7.4 bars), a second period-doubling bifurcation sequence. The apparently chaotic bands near 1 and 7 ms are due to transients and are not of importance here. The second sequence we suggest calling a *subharmonic* period-doubling bifurcation sequence because it is not a window in the chaotic region connected with the first sequence but has its origin in a new attractor of *basic* period $2T_a=2/f_a$ (called a *subharmonic* resonance in nonlinear oscillator theory[9]) which takes over stability from the chaos of basic period T_a and then undergoes itself period doubling to chaos of basic period $2T_a$. Our nomenclature infers that the seemingly similar spectra at about 7.5 and 25 ms are of different origin. This is indeed the case, as can be shown by recourse to the resonant properties of the system, i.e., by tracing back the oscillation in parameter space (f_a,p_a) to the corresponding resonance peak at lower driving pressure amplitudes p_a. It then becomes evident that the second sequence starts from a different resonance than the first one. A full projection of the traced-back bifurcation diagram in (R,f_a,p_a) space cannot be given because of computer-time limitations. As the resonance to which the oscillation at about 25 ms can be traced back is a subharmonic resonance of order $\frac{1}{2}$ (see Ref. 9 for the definition) this oscillation happens to be of doubled period. Depending on the order of the resonance, period tripling, quadrupling, and so on may and does occur (compare Fig. 2 below).

Many spectral bifurcation diagrams have been calculated which lead to the following picture of the bifurcation properties of our model. In the (f_a,p_a) parameter space many period-doubling Feigenbaum direct cascades and Grossmann inverse cascades are encountered. They appear as distinct entities and are therefore called *Feigenbaum-Grossmann objects.* The question of what governs the distribution of these objects in parameter space can be answered by the observation that they adhere to the resonances of the system. The resonances thus impose an ordering on the appearance of Feigenbaum-Grossmann objects, i.e., a distinct bifurcation superstructure.

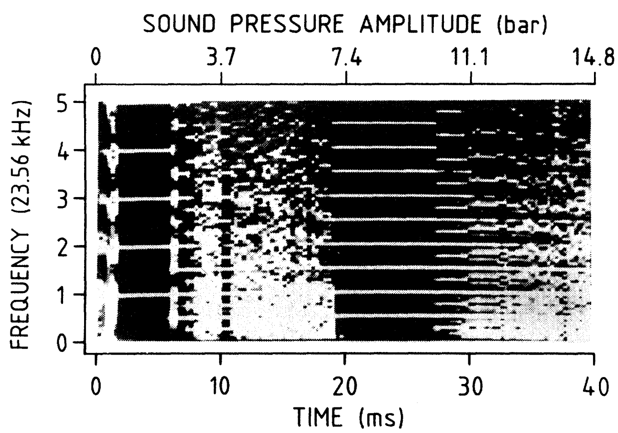

FIG. 1. Spectral bifurcation diagram of bubble wall motion for a bubble with radius at rest of $R_n=100$ μm driven at a frequency of $f_a=23.56$ kHz. The sound pressure amplitude p_a is raised from 0 to 14.8 bars in 40 ms (photograph from a color graphics terminal).

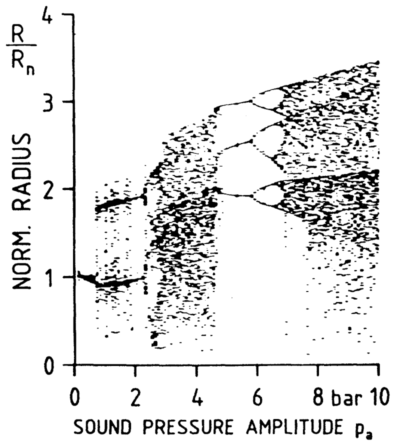

FIG. 2. Bifurcation diagram of a bubble with a radius at rest of $R_n = 230$ μm. Frequency of the driving sound field is $f_a = 23.56$ kHz. The sound pressure amplitude is increased in steps of 0.1 bar. The scattered points that do not follow the obvious pattern are due to transients that have not yet died out (a result of computer-time limitations).

The scenario put forward above is not considered to be a specialty of the above model but to be a general (in the chaos language, universal) feature of most driven nonlinear oscillatory systems. This conjecture gets more support from experiments and calculations with a driven semiconductor oscillator[11] or Toda oscillator.[12] The so-called self-replicating attractor of Brorson, Dewey, and Linsay[11] is conjectured to belong to resonances of basic periods 1, 2, 3, 4, and 5, each with its own separate type of periodic chaos. The driven Toda oscillator[12]

$$\ddot{x} + r\dot{x} + (e^x - 1) = a \cos\omega t \tag{4}$$

is an equation that we believe will become of fundamental importance in the area of nonlinear oscillations as a result of its simplicity. Recently Risken[13] encountered this equation in the context of quantum optics when simplifying a set of laser equations. Equation (4) as well as the experiments[12] show different basic periods and their period doubling to separate types of chaos. One result then is that the bifurcation structure of a driven dissipative nonlinear oscillatory system is composed of many Feigenbaum-Grossmann objects, distributed in parameter space as a kind of elementary fine structure according to the resonance properties of the system in question.

To test our ideas, also ordinary bifurcation diagrams have been calculated via the Poincaré return map. In these diagrams the radii of the bubble in the Poincaré plane of section are plotted versus the sound pressure amplitude. Figure 2 gives a bifurcation diagram for a bubble with a radius at rest of $R_n = 230$ μm in a sound field of frequency $f_a = 23.56$ kHz. First a period-1 oscillation (in units of $T_a = 1/f_a$) is stable; then a jump occurs to a period-2 oscillation. The first subharmonic resonance has taken over before the period-1 oscillation has found its way to chaos and even to the first period-doubling bifurcation. The subharmonic oscillation of basic period 2 then bifurcates by successive period doubling to chaos. Near 5 bars the second subharmonic resonance of basic period 3 takes over, which again bifurcates to chaos. This diagram strongly resembles those of Brorson, Dewey, and Linsay[11] and Klinker, Meyer-Ilse, and Lauterborn,[12] and fits into the resonance scenario developed above.

Next, what we suggest calling *attractor maps* have been calculated as a check of the chaotic properties of the oscillation. As an attractor map we define the map which is given by the approximate curve that may be obtained when one coordinate (say x) of a point on an attractor in a Poincaré plane of section is plotted versus the preceding one for a sufficient number of (usually consecutive) pairs, i.e., $x_{k+1} = \pi_1 P(x_k, y_k)$, where P denotes the Poincaré return map and π_1 the projection on the first entry of P. The first known example of an attractor map is the well-known Lorenz map given by a hatlike function.[14]

It is found that attractor maps may consist of several separate pieces. The number is connected with both the basic period of the attractor from which chaos originated and the inverse cascade of split chaotic regions of Grossmann.[4] In the fully developed chaotic region which occurs after period doubling *and* inverse cascade the number of separate pieces of the attractor map is the number of the basic period from which chaos originated. Thus in the case of the $R_n = 230$-μm bubble and a sound pressure amplitude of 7.4 bars (compare Fig. 2) we get a three-piece attractor map, whereas an $R_n = 100$-μm bubble at 11 bars yields a two-piece attractor map. By the definition of a *subharmonic attractor map of order m* through $x_{k+m} = \pi_1 P^m(x_k, y_k)$, where P^m is the mth iterate of the Poincaré map, and the plotting of x_{k+m} vs x_k, a one-piece map is obtained with a quadraticlike maximum. An example is given in Fig. 3, where r_{k+2} is plotted versus r_k for a 100-μm bubble at 11 bars sound pressure amplitude. Here r is the radius normalized to the maximum radius encountered in the sequence. We have several examples that show that when the inverse cascade is followed in the direc-

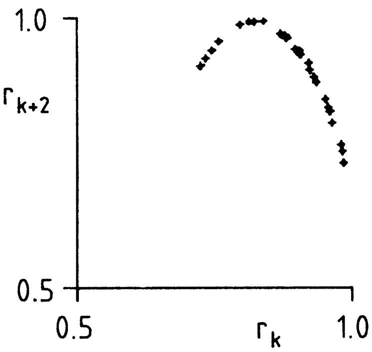

FIG. 3. Subharmonic attractor map of order 2 for the normalized radius $r = R/R_{\max}$, $R_{\max} = 546$ μm. $R_n = 100$ μm, $f_a = 23.56$ kHz, $p_a = 11$ bars, Poincaré plane of section at $\omega t = 155°$.

tion of the splitting of chaotic regions the attractor map also splits, which may be reconciled by using the $(m \times 2^n)$th iterate of the Poincaré map where n is the number of splittings that already occurred in the inverse cascade.

In this paper we have investigated a theoretical model of acoustic turbulence and found a connection between nonlinear resonances and the structure of bifurcation diagrams. This connection is conjectured to be a general (or universal) feature of driven dissipative passive nonlinear oscillatory systems. The nonlinear resonances impose a superstructure on the bifurcation behavior of the system. Feigenbaum direct and Grossman inverse cascades occur as fine structure which may be (partially) suppressed by the gross hysteresis features from the superstructure.

We thank the Fraunhofer-Gesellschaft, Munich, for steady support, and U. Parlitz, W. Meyer-Ilse, A. Kramer, and T. Kurz for valuable discussions. The calculations were done on a UNIVAC 1100 and VAX 11/780 at the GWDG computer center, Göttingen.

[1]Recent reviews are given by R. E. Apfel, in *Ultrasonics,* edited by P. D. Edmonds (Academic, New York, 1981), and by E. A. Neppiras, Phys. Rep. **61**, 159 (1980).

[2]W. Lauterborn and E. Cramer, Phys. Rev. Lett. **47**, 1445 (1981).

[3]M. J. Feigenbaum, J. Stat. Phys. **19**, 25 (1978).

[4]S. Grossman and S. Thomae, Z. Naturforsch. **32a**, 1353 (1977).

[5]See, e.g., *Dynamical Systems and Chaos,* edited by L. Garrido (Springer, Berlin, 1983).

[6]For examples of acoustic cavitation bubble fields, see *Cavitation and Inhomogeneities in Underwater Acoustics,* edited by W. Lauterborn (Springer, New York, 1980).

[7]F. R. Gilmore, California Institute of Technology Report No. 26-4, 1952 (unpublished). For a more accessible derivation, see *High Intensity Ultrasonic Fields,* edited by L. D. Rozenberg (Plenum, New York, 1971).

[8]E. Cramer and W. Lauterborn, Acustica **49**, 226 (1981).

[9]W. Lauterborn, J. Acoust. Soc. Am. **59**, 283 (1976).

[10]W. Lauterborn, Appl. Sci. Res. **38**, 165 (1982).

[11]S. D. Brorson, D. Dewey, and P. S. Linsay, Phys. Rev. A **28**, 1201 (1983).

[12]T. Klinker, W. Meyer-Ilse, and W. Lauterborn, Phys. Lett. **101A**, 371 (1984).

[13]H. Risken, private communication.

[14]E. N. Lorenz, J. Atmos. Sci. **20**, 130 (1963).

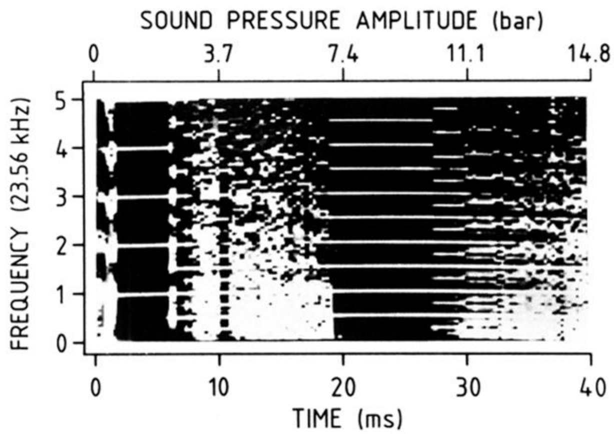

FIG. 1. Spectral bifurcation diagram of bubble wall motion for a bubble with radius at rest of $R_n = 100\ \mu$m driven at a frequency of $f_a = 23.56$ kHz. The sound pressure amplitude p_a is raised from 0 to 14.8 bars in 40 ms (photograph from a color graphics terminal).