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Simulation of a Critical Ising Fractal

Gyan Bhanot[a]
The Institute for Advanced Study, Princeton, New Jersey 08540

and

Herbert Neuberger
Brookhaven National Laboratory, Upton, New York 11973, and
Department of Physics and Astronomy, Rutgers University,[b] Piscataway, New Jersey 08854

and

Joel A. Shapiro
Department of Physics and Astronomy, Rutgers University,
Piscataway, New Jersey 08854
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The results of microcanonical simulation of an Ising model defined on a truncated, lattice-embedded, fractal, Sierpinsky carpet of dimension ~ 1.86 are presented. Estimates for a variety of exponents give, via hyperscaling at criticality and via the behavior of the spin-spin correlation away from criticality, consistently, a dimension $\sim 1.66 \pm 0.10$.

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A most attractive property of fractals[1] is that they possess a well defined noninteger dimension. Noninteger dimensions also arise in the study of renormalization-group equations[2] where the definition of the continuation to arbitrary dimensions is restricted to perturbation theory. Recently, it was suggested that it might be possible to interpolate between integer dimensions by lattice embedment of a special type of fractal with Ising spins on its sites.[3] The authors of Ref. 3 pointed out that a central assumption when continuing in dimension in perturbation theory is that translation invariance is respected. Therefore they considered a subclass of fractals for which there is a limit where translation invariance is restored. In this limit, the dependence on all geometrical characteristics of the fractal, with the exception of the dimension, disappears and it was conjectured that a continuation in dimension which is compatible with the one given by perturbation theory is obtained. The basis for this conjecture was a comparison between results to $O(\epsilon)$ $(d=1+\epsilon)$ obtained in perturbation theory[4] and a Migdal-Kadanoff decimation.[5]

The conjecture certainly deserves further analysis, especially in view of the possibility of generalizing it to lattice gauge systems in any dimensions. In particular we have in mind the simulation of a gauge system in $4-\epsilon$ dimensions, $\epsilon \ll 1$. The present authors are unaware of any detailed numerical studies of the critical properties of fractal Ising models.[6] One possible reason is that the special characteristics of a fractal system preclude a

straightforward application of standard numerical techniques without rather long simulation times.

Our main purpose in this Letter is to show that, by employing the recently proposed microcanonical simulation methods,[7] numerical studies of critical properties on fractals is possible with even modest computer facilities. We shall report below estimates of a variety of critical exponents which were obtained from simulations of a total of about 80 CPU hours of VAX 11/780. Our results indicate that for practical purposes, a self-consistent picture emerges even for numerically manageable lattices which violate translation invariance rather badly.

We used a fractal of the Sierpinsky carpet type,[1] embedded in a 64×64 lattice. It is constructed by repeated applications of the following procedure: Take a square and divide it into $b\times b$ parts. Divide each of these again into $c\times c$ parts and cut out a central $l\times l$ section. For our simulation we used a lattice where this procedure was carried out twice with $b=2$, $c=4$, and $l=2$. On the sites of the fractal live Ising spins coupled with strength β_i along internal links and strength β_b along the numerous boundary links. Periodic boundary conditions are imposed. The partition function is given by

$$Z=\sum_{[s]}\exp\left(\beta_i\sum_{\substack{\text{internal}\\\text{links}}}s_{\vec{n}}s_{\vec{n}+\hat{e}}+\beta_b\sum_{\substack{\text{boundary}\\\text{links}}}s_{\vec{n}}s_{\vec{n}+\hat{e}}\right). \tag{1}$$

The fractal dimension D of the system, in the limit of infinite divisions, is $D=\ln48/\ln8=1.86\ldots$. Because of the facts that D is far from unity and the decimation factor $bc=8$ is rather large, one expects that the Migdal-Kadanoff recursion relation will work rather poorly on a quantitative level.

Our microcanonical simulation involves two sets of "demons," one carrying energy E_i^d conjugate to β_i and the other carrying energy E_b^d conjugate to β_b. A local (assumed ergodic) deterministic evolution is defined which conserves separately E_i^T and E_b^T defined by

$$E_i^T=E_i^d+\sum_{\substack{\text{internal}\\\text{links}}}s_{\vec{n}}s_{\vec{n}+\hat{e}},\qquad E_b^T=E_b^d+\sum_{\substack{\text{boundary}\\\text{links}}}s_{\vec{n}}s_{\vec{n}+\hat{e}}. \tag{2}$$

With microcanonical updating in assembly language, we were able to obtain a speed of 300 000 full lattice updates per VAX 11/780 CPU hour.[7] Our simulation routines were checked against other, transparent, routines written all in Fortran. The couplings β_i and β_b were extracted from the distribution of the demon energies. Since we were close to two dimensions, we checked all our methods on the two-dimensional model.

We started with a few relatively short runs to approximately map out the phase diagram. This is shown in Fig. 1 and is in qualitative agreement with what one expects.[3] To compute critical exponents, we selected a line L (see Fig. 1) and our subsequent runs were made on points close to L. One cannot put these points exactly on L because one controls β_i and β_b only through E_i^T and E_b^T. At each point we made 600 000 sweeps with 100 000 measurements to avoid some of the correlations. The accuracy of the β_i,β_b determination is of order 0.1% and the dispersion of the points around L is of a similar magnitude.

We check the convergence of the simulation by comparing the measured internal energy with a strong-coupling expansion of the partition function to order $\tanh^4\beta_i$.

For reasons of program speed, we only measured on-axis two-point correlations on lines which encountered no holes. The correlations were fitted by a form

$$\langle s_{\vec{0}}s_{\vec{n}}\rangle_{\text{on axis}}=A(\beta_i)\left(\frac{\exp[-m(\beta_i)|\vec{n}|]}{|\vec{n}|^{\rho(\beta_i)}}+\frac{\exp[-m(\beta_i)(64-|\vec{n}|)]}{(64-|\vec{n}|)^{\rho(\beta_i)}}\right) \tag{3}$$

for various β_i's on the line L in the symmetric phase [$m(\beta_i)$ is the inverse correlation length]. Fitting to a region for which $m(\beta_i)|\vec{n}|\gtrsim1$, $\rho(\beta_i)=\frac{1}{2}(d_s-1)$, where d_s is another type of dimension, possibly related to the spectral dimension.[8] For β_i near β_i^c, $m(\beta_i)\sim m_0|\beta_i^c-\beta_i|^\nu$. In the region of large correlation lengths, a fit over $\langle s_{\vec{0}}s_{\vec{n}}\rangle$, with $m(\beta_i)|\vec{n}|\ll1$, is insensitive to m and $\rho(\beta_i)=\Delta_\phi=d^c-2+\eta$, where d^c is the dimension to be associated with hyperscaling. We decided at which β_i values to take data and make fits by using exactly known facts about the two-dimensional Ising model as guidelines. Our numerical results are

$$\beta_i^c=0.511\pm0.003,\quad \nu=1.28\pm0.05,\qquad \Delta_\phi=0.20\pm0.02,\quad d_s=1.70\pm0.10. \tag{4}$$

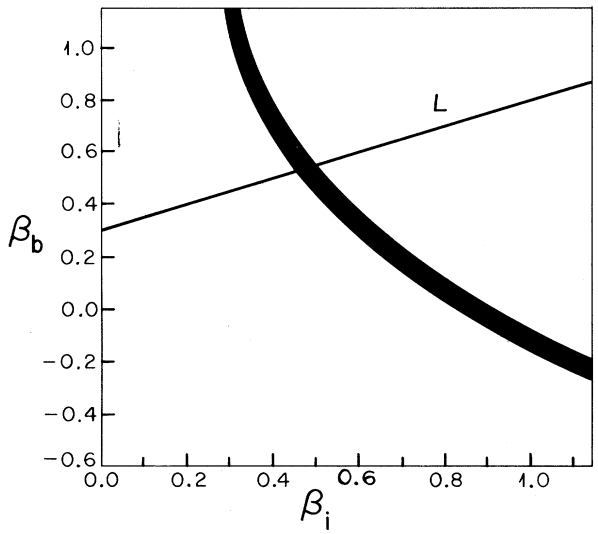

FIG. 1. Phase diagram in the β_i, β_b plane. The solid band is approximate location of continuous transitions. The line labeled L is the one along which a detailed study for critical exponents was done.

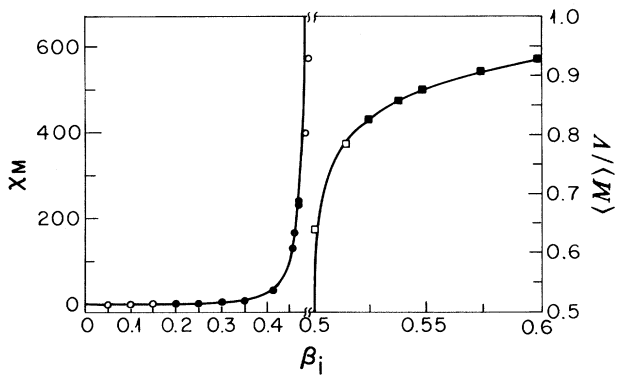

FIG. 2. Plot of χ_M [see Eq. (5a)] and $\langle M \rangle / V$ [see Eq. (6a)] vs β_i for simulation along the line L of Fig. 1. The fits are shown as solid curves. The fits were only made for the points shown as solid dots. Note the change of scale at $\beta_i = 0.5$. Also note the different scales for the two quantities plotted.

Fluctuations in the magnetization of the microcanonical system are equal to those of a canonical system with the appropriate couplings, even at infinite volume, only in the disordered phase. Our final estimates from a variety of fits of the type

$$\chi_M = V^{-1} \langle (\textstyle\sum_{\vec{n}} s_{\vec{n}})^2 \rangle_{\text{disord}} \sim A \, |\beta_i - \beta_i^c|^{-\gamma} \tag{5a}$$

gave

$$\beta_i^c = 0.500 \pm 0.005, \quad \gamma = 1.90 \pm 0.05. \tag{5b}$$

The last exponent we measured was $\tilde{\beta}$, which is associated with the spontaneous magnetization in the broken phase,

$$\langle M \rangle / V = V^{-1} \langle \textstyle\sum_{\vec{n}} s_{\vec{n}} \rangle \sim A \, |\beta_i - \beta_i^c|^{\tilde{\beta}}. \tag{6a}$$

We obtained

$$\beta_i^c = 0.498 \pm 0.005, \quad \tilde{\beta} = 0.10 \pm 0.02. \tag{6b}$$

The data used to find the magnetic exponents and the respective fits are shown in Fig. 2.

To check our methods, we also carried out microcanonical simulations of comparable accuracy on a 64×64 two-dimensional Ising model and analyzed the data with identical procedures. The results obtained agreed with the exact exponents of the two-dimensional model with error bars similar to those in Eqs. (4)–(6). All these errors come from a variety of sources: statistical fluctuations, variations among various types of fits to the background (subleading corrections to the scaling forms), and dependences on cuts. However, it is fair to say that our error estimates are only qualitative.

Assuming that the system at criticality is governed by a single dimension d^c, we get two independent estimates for it:

$$d^c = \gamma + 2\tilde{\beta}/\nu = 1.64 \pm 0.08 \tag{7a}$$

and

$$d^c = \Delta_\phi + \gamma/\nu = 1.68 \pm 0.07. \tag{7b}$$

These two numbers agree with each other (however, this mostly reflects the smallness of Δ_ϕ and $\tilde{\beta}/\nu$). They also agree with the dimension d_s [see Eq. (4)].

It is not obvious *a priori* whether d_s and d^c should come out equal as long as the system is not close to the limit of translational invariance. The fact that both d^c and d_s are systematically shifted from the fractal dimension D could be a reflection of the violation of translation invariance and the finiteness of the volume of our system. One can define a variety of "fractal dimensions" by looking at how the numbers of internal links, boundary links, internal sites, and boundary sites scale between our system and one embedded in an 8×8 lattice. The numbers obtained range from 1.81 to 1.94, the lowest figure coming from internal sites; none of these numbers is in good agreement with Eqs. (7). Other estimates we have tried gave an even smaller range of values for the "fractal dimension."[9]

The use of special on-axis correlations might also

mean that ν, d^c, and d_s have (unknown) systematic errors. The exponent ν could be better estimated by looking at correlations on an 8×8 lattice representing 8×8 blocks of our original lattice. Indeed, it was in order to keep this possibility open that we picked $b=2$, $c=4$, and $l=2$. The existing program is able to measure such blocked correlations. Finite-size effects are also a source of systematic errors. To establish these numerically one needs to go to a system size 512^2 which is barely accessible within our methods and prohibitively large in other approaches.

It is clear from this study that detailed numerical studies of critical phenomena on fractals of infinite ramification are quite feasible. The observed discrepancy between d^c or d_s and D, if upheld by more extensive studies, requires an explanation.

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[a]Present address: Institute for Theoretical Physics, University of California at Santa Barbara, Santa Barbara, Cal. 93106.

[b]Permanent address.

[1]B. B. Mandelbrot, *Fractals: Form, Chance and Dimension* (Freeman, San Francisco, 1977), and *The Fractal Geometry of Nature* (Freeman, San Francisco, 1982).

[2]K. G. Wilson and M. E. Fisher, Phys. Rev. Lett. **28**, 240 (1983).

[3]Y. Gefen, Y. Meir, B. B. Mandelbrot, and A. Aharony, Phys. Rev. Lett. **50**, 145 (1983).

[4]D. J. Wallace and R. K. P. Zia, Phys. Rev. Lett. **43**, 808 (1981).

[5]A. A. Migdal, Zh. Eksp. Teor. Fiz. **69**, 1457 (1975) [Sov. Phys. JETP **42**, 743 (1975)]; L. P. Kadanoff, Ann. Phys. (N.Y.) **100**, 359 (1976).

[6]See, however, the paper by D. Andelman, H. Orland, and L. C. R. Wijewardhana, Phys. Rev. Lett. **52**, 145 (1984). These authors looked for the ordering transition in fractal Ising systems in random external fields.

[7]G. Bhanot, M. Creutz, and H. Neuberger, Nucl. Phys. **B235** [FS11], 417 (1984). A two-dimensional 64^2 Ising system is simulated at 2×10^6 lattice updates per hour.

[8]See, for example, R. Rammal and G. Toulouse, J. Phys. (Paris), Lett. **44**, L13 (1983).

[9]On an $8^k\times 8^k$ lattice, the number of internal links is $8^k\times(8\times 6^k+2)/5$, the number of internal sites is $(121\times 48^k+94\times 8^k+20)/235$, and the number of boundary links and sites are each $4\times 8^k\times(6^k-1)/5$. For our 64×64 lattice these give estimates for the dimension of 1.8557, 1.8433, and 1.8863, respectively, compared to the infinite-lattice dimension of 1.8617.